A Primer of Real Analysis

Dan Sloughter
Department of Mathematics
Furman University

November 24, 2008

Preface

This is a short introduction to the fundamentals of real analysis. Although the prerequisites are few, I have written the text assuming the reader has the level of mathematical maturity of one who has completed the standard sequence of calculus courses, has had some exposure to the ideas of mathematical proof (including induction), and has an acquaintance with such basic ideas as equivalence relations and the elementary algebraic properties of the integers.

Contents

Chapter 1

Fundamentals

1.1 Sets and relations

1.1.1 The integers

Kronecker once said, "God made the integers; all the rest is the work of man." Taking this as our starting point, we assume the existence of the set

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}, \tag{1.1.1}$$

the set of integers. Moreover, we assume the properties of the operations of addition and multiplication of integers, along with other elementary properties such as the *Fundamental Theorem of Arithmetic*, that is, the statement that every integer may be factored into a product of prime numbers and this factorization is essentially unique.

1.1.2 Sets

We will take a naive view of sets: given any property p, we may determine a set by collecting together all objects which have property p. This may be done either by explicit enumeration, such as, p is the property of being one of a, b, or c, which creates the set $\{a, b, c\}$, or by stating the desired property, such as, p is the property of being a positive integer, which creates the set

$$\mathbb{Z}^+ = \{1, 2, 3, 4, \ldots\}. \tag{1.1.2}$$

The notation $x \in A$ indicates that x is an element of the set A. Given sets A and B, we say A is a *subset* of B, denoted $A \subset B$, if from the fact that $x \in A$ it necessarily follows that $x \in B$. We say sets A and B are *equal* if both $A \subset B$ and $B \subset A$.

Given two sets A and B, we call the set

$$A \cup B = \{x : x \in A \text{ or } x \in B\} \tag{1.1.3}$$

the *union* of A and B and the set

$$A \cap B = \{x : x \in A \text{ and } x \in B\} \tag{1.1.4}$$

the *intersection* of A and B. We call the set

$$A \setminus B = \{x : x \in A, x \notin B\} \tag{1.1.5}$$

the *difference* of A and B.

More generally, if I is a set and $\{A_\alpha : \alpha \in I\}$ is a collection of sets, one for each element of I, then we have the union

$$\bigcup_{\alpha \in I} A_\alpha = \{x : x \in A_\alpha \text{ for some } \alpha\} \tag{1.1.6}$$

and the intersection

$$\bigcap_{\alpha \in I} A_\alpha = \{x : x \in A_\alpha \text{ for all } \alpha\}. \tag{1.1.7}$$

Example 1.1.1. For example, if $I = \{2, 3, 4, \ldots\}$ and we let

$$A_2 = \{n : n \in \mathbb{Z}^+, n > 1, n \neq 2m \text{ for any } m \in \mathbb{Z}^+ \text{ with } m > 1\}$$

and, for any $i \in I$, $i > 2$,

$$A_i = \{n : n \in A_{i-1}, n \neq mi \text{ for any } m \in \mathbb{Z}^+ \text{ with } m > 1\},$$

then $\bigcap_{i \in I} A_i$ is the set of prime numbers.

If A and B are both sets, we call the set

$$A \times B = \{(a, b) : a \in A, b \in B\} \tag{1.1.8}$$

the *cartesian product* of A and B. If $A = B$, we write

$$A^2 = A \times A. \tag{1.1.9}$$

Example 1.1.2. $\mathbb{Z}^2 = \{(m, n) : m \in \mathbb{Z}, n \in \mathbb{Z}\}$ is the set of all ordered pairs of integers.

1.1.3 Relations

Given two sets A and B, we call a subset R of $A \times B$ a *relation*. Given a relation R, we will write $a \sim_R b$, or simply $a \sim b$ if R is clear from the context, to indicate that $(a, b) \in R$.

Example 1.1.3. We say that an integer m *divides* and integer n if $n = mi$ for some integer i. If we let

$$R = \{(m, n) : m \in \mathbb{Z}, n \in \mathbb{Z}, m \text{ divides } n\},$$

then R is a relation. For example, $3 \sim_R 12$.

Consider a set A and a relation $R \subset A^2$. For purposes of conciseness, we say simply that R is a relation *on* A. If R is such that $a \sim_R a$ for every $a \in A$, we say R is *reflexive*; if R is such that $b \sim_R a$ whenever $a \sim_R b$, we say R is *symmetric*; if $a \sim_R b$ and $b \sim_R c$ together imply $a \sim_R c$, we say R is *transitive*. We call a relation which is reflexive, symmetric, and transitive an *equivalence relation*.

Exercise 1.1.1. Show that the relation R on $\mathbb{Z}$ defined by $m \sim_R n$ if m divides n is reflexive and transitive, but not symmetric.

Exercise 1.1.2. Show that the relation R on $\mathbb{Z}$ defined by $m \sim_R n$ if $m - n$ is even is an equivalence relation.

Given an equivalence relation R on a set A and an element $x \in A$, we call

$$[x] = \{y : y \in A, y \sim_R x\} \tag{1.1.10}$$

the *equivalence class* of x.

Exercise 1.1.3. Given an equivalence relation R on a set A, show that

a. $[x] \cap [y] \neq \emptyset$ if and only if $x \sim_R y$;

b. $[x] = [y]$ if and only if $x \sim_R y$.

As a consequence of the previous exercise, the equivalence classes of an equivalence relation on a set A constitute a *partition* of A (that is, A may be written as the disjoint union of the equivalence classes).

Exercise 1.1.4. Find the equivalence classes for the equivalence relation in Exercise 1.1.2.

1.2 Functions

If A and B are sets, we call a relation $R \subset A \times B$ a *function* with *domain* A if for every $a \in A$ there exists one, and only one, $b \in B$ such that $(a, b) \in R$. We typically indicate such a relation with the notation $f : A \to B$, and write $f(a) = b$ to indicate that $(a, b) \in R$. We call the set of all $b \in B$ such that $f(a) = b$ for some $a \in A$ the *range* of f. With this notation, we often refer to R as the *graph* of f.

We say $f : A \to B$ is *one-to-one* if for every b in the range of f there exists a unique $a \in A$ such that $f(a) = b$. We say f is *onto* if for every $b \in B$ there exists at least one $a \in A$ such that $f(a) = b$. For example, the function $f : \mathbb{Z}^+ \to \mathbb{Z}^+$

defined by $f(z) = z^2$ is one-to-one, but not onto, whereas the function $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(z) = z + 1$ is both one-to-one and onto.

Given two functions, $g : A \to B$ and $f : B \to C$, we define the *composition*, denoted $f \circ g : A \to C$, to be the function defined by $f \circ g(a) = f(g(a))$.

If $f : A \to B$ is both one-to-one and onto, then we may define a function $f^{-1} : B \to A$ by requiring $f^{-1}(b) = a$ if and only if $f(a) = b$. Note that this implies that $f \circ f^{-1}(b) = b$ for all $b \in B$ and $f^{-1} \circ f(a) = a$ for all $a \in A$. We call f^{-1} the *inverse* of f.

Given any collection of nonempty sets, $\{A_\alpha\}$, $\alpha \in I$, we assume the existence of a function $\phi : I \to B$, $B = \bigcup_{\alpha \in I} A_\alpha$, with the property that $\phi(\alpha) \in A_\alpha$. We call such a function a *choice function*. The assumption that choice functions always exist is known as the *Axiom of Choice*.

1.3 Rational numbers

Let $P = \{(p, q) : p, q \in \mathbb{Z}, q \neq 0\}$. We define an equivalence relation on P by saying $(p, q) \sim (s, t)$ if $pt = qs$.

Exercise 1.3.1. Show that the relation as just defined is indeed an equivalence relation.

We will denote the equivalence class of $(p, q) \in P$ by p/q, or $\frac{p}{q}$. We call the set of all equivalence classes of P the *rational numbers*, which we denote by $\mathbb{Q}$. If $p \in \mathbb{Z}$, we will denote the equivalence class of $(p, 1)$ by p; that is, we let

$$\frac{p}{1} = p. \tag{1.3.1}$$

In this way, we may think of $\mathbb{Z}$ as a subset of $\mathbb{Q}$.

1.3.1 Field properties

We wish to define operations of addition and multiplication on elements of $\mathbb{Q}$. We begin by defining operations on the elements of P. Namely, given $(p, q) \in P$ and $(s, t) \in P$, define

$$(p, q) \oplus (s, t) = (pt + sq, qt) \tag{1.3.2}$$

and

$$(p, q) \otimes (s, t) = (ps, qt). \tag{1.3.3}$$

Now suppose $(p, q) \sim (a, b)$ and $(s, t) \sim (c, d)$. It follows that $(p, q) \oplus (s, t) \sim (a, b) \oplus (c, d)$, that is, $(pt + sq, qt) \sim (ad + cb, bd)$, since

$$(pt + sq)bd = pbtd + sdqb = qatd + tcqb = (ad + cb)qt. \tag{1.3.4}$$

Moreover, $(p, q) \otimes (s, t) \sim (a, b) \otimes (c, d)$, that is, $(ps, qt) \sim (ac, bd)$, since

$$psbd = pbsd = qatc = qtac. \tag{1.3.5}$$

This shows that the equivalence class of a sum or product depends only on the equivalence classes of the elements being added or multiplied. Thus we may define addition and multiplication on $\mathbb{Q}$ by

$$\frac{p}{q} + \frac{s}{t} = \frac{pt + sq}{qt} \tag{1.3.6}$$

and

$$\frac{p}{q} \times \frac{s}{t} = \frac{ps}{qt}, \tag{1.3.7}$$

and the results will not depend on which representatives we choose for each equivalence class. Of course, multiplication is often denoted using juxtaposition, that is,

$$\frac{p}{q} \times \frac{s}{t} = \frac{p}{q}\frac{s}{t}, \tag{1.3.8}$$

and repeated multiplication may be denoted by exponentiation, that is, a^n, $a \in \mathbb{Q}$ and $n \in \mathbb{Z}^+$, represents the product of a with itself n times.

Note that if $(p, q) \in P$, then $(-p, q) \sim (p, -q)$. Hence, if $a = \frac{p}{q} \in \mathbb{Q}$, then we let

$$-a = \frac{-p}{q} = \frac{p}{-q}. \tag{1.3.9}$$

For any $a, b \in \mathbb{Q}$, we will write $a - b$ to denote $a + (-b)$.

If $a = \frac{p}{q} \in \mathbb{Q}$ with $p \neq 0$, then we let

$$a^{-1} = \frac{q}{p}. \tag{1.3.10}$$

Moreover, we will write

$$\frac{1}{a} = a^{-1}, \tag{1.3.11}$$

$$\frac{1}{a^n} = a^{-n} \tag{1.3.12}$$

for any $n \in \mathbb{Z}^+$, and, for any $b \in \mathbb{Q}$,

$$\frac{b}{a} = ba^{-1}. \tag{1.3.13}$$

It is now easy to show that

1. $a + b = b + a$ for all $a, b \in \mathbb{Q}$;
2. $(a + b) + c = a + (b + c)$ for all $a, b, c \in \mathbb{Q}$;
3. $ab = ba$ for all $a, b \in \mathbb{Q}$;
4. $(ab)c = a(bc)$ for all $a, b, c \in \mathbb{Q}$;
5. $a(b + c) = ab + ac$ for all $a, b, c \in \mathbb{Q}$;
6. $a + 0 = a$ for all $a \in \mathbb{Q}$;

7. $a + (-a) = 0$ for all $a \in \mathbb{Q}$;

8. $1a = a$ for all $a \in \mathbb{Q}$;

9. if $a \in \mathbb{Q}$, $a \neq 0$, then $aa^{-1} = 1$.

Taken together, these statements imply that $\mathbb{Q}$ is a *field*.

1.3.2 Order and metric properties

We say a rational number a is *positive* if there exist $p, q \in \mathbb{Z}^+$ such that $a = \frac{p}{q}$. We denote the set of all positive elements of $\mathbb{Q}$ by $\mathbb{Q}^+$.

Given $a, b \in \mathbb{Q}$, we say a is *less than* b, or, equivalently, b is *greater than* a, denoted either by $a < b$ or $b > a$, if $b - a$ is positive. In particular, $a > 0$ if and only if a is positive. If $a < 0$, we say a is *negative*. We write $a \leq b$, or, equivalently, $b \geq a$, if either $a < b$ or $a = b$.

Exercise 1.3.2. Show that for any $a \in \mathbb{Q}$, one and only one of the following must hold: (a) $a < 0$, (b) $a = 0$, (c) $a > 0$.

Exercise 1.3.3. Show that if $a, b \in \mathbb{Q}^+$, then $a + b \in \mathbb{Q}^+$.

Exercise 1.3.4. Suppose $a, b, c \in \mathbb{Q}$. Show each of the following:

a. One, and only one, of the following must hold:

   (i) $a < b$,

   (ii) $a = b$,

   (iii) $a > b$.

b. If $a < b$ and $b < c$, then $a < c$.

c. If $a < b$, then $a + c < b + c$.

d. If $a > 0$ and $b > 0$, then $ab > 0$.

Exercise 1.3.5. Show that if $a, b \in \mathbb{Q}$ with $a > 0$ and $b < 0$, then $ab < 0$.

Exercise 1.3.6. Show that if $a, b, c \in \mathbb{Q}$ with $a < b$, then $ac < bc$ if $c > 0$ and $ac > bc$ if $c < 0$.

Exercise 1.3.7. Show that if $a, b \in \mathbb{Q}$ with $a < b$, then

$$a < \frac{a+b}{2} < b.$$

As a consequence of Exercise 1.3.4 we say $\mathbb{Q}$ is an *ordered field*.

For any $a \in \mathbb{Q}$, we call

$$|a| = \begin{cases} a, & \text{if } a \geq 0, \\ -a, & \text{if } a < 0, \end{cases} \tag{1.3.14}$$

the *absolute value* of a.

Exercise 1.3.8. Show that for any $a \in \mathbb{Q}$, $-|a| \leq a \leq |a|$.

Proposition 1.3.1. For any $a, b \in \mathbb{Q}$, $|a+b| \leq |a| + |b|$.

Proof. If $a + b \geq 0$, then

$$|a| + |b| - |a+b| = |a| + |b| - a - b = (|a| - a) + (|b| - b). \tag{1.3.15}$$

Both of the terms on the right are nonnegative by Exercise 1.3.8. Hence the sum is nonnegative and the proposition follows. If $a + b < 0$, then

$$|a| + |b| - |a+b| = |a| + |b| + a + b = (|a| + a) + (|b| + b). \tag{1.3.16}$$

Again, both of the terms on the right are nonnegative by Exercise 1.3.8. Hence the sum is nonnegative and the theorem follows. **Q.E.D.**

It is now easy to show that the absolute value satisfies

1. $|a - b| \geq 0$ for all $a, b \in \mathbb{Q}$, with $|a - b| = 0$ if and only if $a = b$,

2. $|a - b| = |b - a|$ for all $a, b \in \mathbb{Q}$,

3. $|a - b| \leq |a - c| + |c - b|$ for all $a, b, c \in \mathbb{Q}$.

Note that the last statement, known as the *triangle inequality*, follows from writing

$$a - b = (a - c) + (c - b) \tag{1.3.17}$$

and applying the previous proposition. These properties show that the function

$$d(a, b) = |a - b| \tag{1.3.18}$$

is a *metric*, and we will call $|a - b|$ the *distance* from a to b.

Suppose $a, b \in \mathbb{Q}^+$ with $a < b$ and let $p, q, r, s \in \mathbb{Z}^+$ such that $a = \frac{p}{q}$ and $b = \frac{r}{s}$. For any $n \in \mathbb{Z}^+$, we have

$$na - b = n\frac{p}{q} - \frac{r}{s} = \frac{nps - rq}{qs}. \tag{1.3.19}$$

If we choose n large enough so that $nps - rq > 0$, it follows that $na - b > 0$, that is, $na > b$. We say that the ordered field $\mathbb{Q}$ is *archimedean*. Note that it also follows that we may choose n large enough to ensure that $\frac{b}{n} < a$.

1.3.3 Upper and lower bounds

Definition 1.3.1. Let $A \subset \mathbb{Q}$. If $s \in \mathbb{Q}$ is such that $s \geq a$ for every $a \in A$, then we call s an *upper bound* for A. If s is an upper bound for A with the property that $s \leq t$ whenever t is an upper bound for A, then we call s the *supremum*, or *least upper bound*, of A, denoted $s = \sup A$. Similarly, if $r \in \mathbb{Q}$ is such that $r \leq a$ for every $a \in A$, then we call r a *lower bound* for A. If r is a lower bound for A with the property that $r \geq t$ whenever t is a lower bound for A, then we call r the *infimum*, or *greatest lower bound*, of A, denoted $r = \inf A$.

Exercise 1.3.9. Show that the supremum of a set $A \subset \mathbb{Q}$, if it exists, is unique, and thus justify the use of the definite article in the previous definition.

A set which does not have an upper bound will not, *a fortiori*, have a supremum. Moreover, even sets which have upper bounds need not have a supremum.

Example 1.3.1. $\mathbb{Q}$ does not have an upper bound.

Example 1.3.2. Consider the set

$$A = \{a : a \in \mathbb{Q}^+, a^2 < 2\}.$$

Note that if $a, b \in \mathbb{Q}^+$ with $a < b$, then

$$b^2 - a^2 = (b-a)(b+a) > 0,$$

from which it follows that $a^2 < b^2$. Hence if $a \in \mathbb{Q}^+$ with $a^2 > 2$, then a is an upper bound for A. For example, 4 is an upper bound for A.

Now suppose $s \in \mathbb{Q}^+$ is the supremum of A. We must have either $s^2 < 2$, $s^2 > 2$, or $s^2 = 2$.

Suppose $s^2 < 2$ and let $\epsilon = 2 - s^2$. By the archimedean property of $\mathbb{Q}$, we may choose $n \in \mathbb{Z}^+$ such that

$$\frac{2s+1}{n} < \epsilon,$$

from which it follows that

$$\frac{2s}{n} + \frac{1}{n^2} = \frac{2s + \frac{1}{n}}{n} \leq \frac{2s+1}{n} < \epsilon.$$

Hence

$$\left(s + \frac{1}{n}\right)^2 = s^2 + \frac{2s}{n} + \frac{1}{n^2} < s^2 + \epsilon = 2,$$

which implies that $s + \frac{1}{n} \in A$. Since $s < s + \frac{1}{n}$, this contradicts the assumption that s is an upper bound for A.

So now suppose $s^2 > 2$. Again let $n \in \mathbb{Z}^+$ and note that

$$\left(s - \frac{1}{n}\right)^2 = s^2 - \frac{2s}{n} + \frac{1}{n^2}.$$

If we let $\epsilon = s^2 - 2$, then we may choose $n \in \mathbb{Z}^+$ so that

$$\frac{2s}{n} < \epsilon.$$

It follows that

$$\left(s - \frac{1}{n}\right)^2 > s^2 - \epsilon + \frac{1}{n^2} = 2 + \frac{1}{n^2} > 2.$$

Thus $s - \frac{1}{n}$ is an upper bound for A and $s - \frac{1}{n} < s$, contradicting the assumption that $s = \sup A$.

Thus we must have $s^2 = 2$. However, this is impossible in light of the following proposition. Hence we must conclude that A does not have a supremum.

Proposition 1.3.2. There does not exist a rational number s with the property that $s^2 = 2$.

Proof. Suppose there exists $s \in \mathbb{Q}$ such that $s^2 = 2$. Choose $a, b \in \mathbb{Z}^+$ so that a and b are relatively prime (that is, they have no factor other than 1 in common) and $s = \frac{a}{b}$. Then

$$\frac{a^2}{b^2} = 2,$$

so $a^2 = 2b^2$. Thus a^2, and hence a, is an even integer. So there exists $c \in \mathbb{Z}^+$ such that $a = 2c$. Hence

$$2b^2 = a^2 = 4c^2,$$

from which it follows that $b^2 = 2c$, and so b is also an even integer. But this contradicts the assumption that a and b are relatively prime. **Q.E.D.**

Exercise 1.3.10. Show that there does not exist a rational number s with the property that $s^2 = 3$.

Exercise 1.3.11. Show that there does not exist a rational number s with the property that $s^2 = 6$.

Exercise 1.3.12. Let $A = \{a : a \in \mathbb{Q}, a^3 < 2\}$.

1. Show that if $a \in A$ and $b < a$, then $b \in A$.

2. Show that if $a \notin A$, and $b > a$, then $b \notin A$.

1.3.4 Sequences

Definition 1.3.2. Suppose $n \in \mathbb{Z}$, $I = \{n, n+1, n+2, \ldots\}$, and A is a set. We call a function $\varphi : I \to A$ a *sequence* with values in A.

Frequently, we will define a sequence φ by specifying its values with notation such as, for example, $\{\varphi(i)\}_{i\in I}$, or $\{\varphi(i)\}_{i=n}^{\infty}$. Thus, for example, $\{i^2\}_{i=1}^{\infty}$ denotes the sequence $\varphi : \mathbb{Z}^+ \to \mathbb{Z}$ defined by $\varphi(i) = i^2$. Moreover, it is customary to denote the values of a sequence using subscript notation. Thus if $a_i = \varphi(i)$, $i \in I$, then $\{a_i\}_{i\in I}$ denotes the sequence φ. For example, we may define the sequence of the previous example by writing $a_i = i^2$, $i = 1, 2, 3, \ldots$.

Definition 1.3.3. Suppose $\{a_i\}_{i\in I}$ is a sequence with values in $\mathbb{Q}$. We say that $\{a_i\}_{i\in I}$ *converges*, and has *limit* L, $L \in \mathbb{Q}$, if for every $\epsilon \in \mathbb{Q}^+$, there exists $N \in \mathbb{Z}$ such that

$$|a_i - L| < \epsilon \text{ whenever } i > N. \tag{1.3.20}$$

If the sequence $\{a_i\}_{i\in I}$ converges to L, we write

$$\lim_{i\to\infty} a_i = L. \tag{1.3.21}$$

Example 1.3.3. We have

$$\lim_{i\to\infty} \frac{1}{i} = 0,$$

since, for any rational number $\epsilon > 0$,

$$\left|\frac{1}{i} - 0\right| = \frac{1}{i} < \epsilon$$

for any $i > N$, where N is any integer larger than $\frac{1}{\epsilon}$.

Definition 1.3.4. Suppose $\{a_i\}_{i\in I}$ is a sequence with values in $\mathbb{Q}$. We call $\{a_i\}_{i\in I}$ a *Cauchy sequence* if for every $\epsilon \in \mathbb{Q}^+$, there exists $N \in \mathbb{Z}$ such that

$$|a_i - a_k| < \epsilon \text{ whenever both } i > N \text{ and } k > N. \tag{1.3.22}$$

Proposition 1.3.3. If $\{a_i\}_{i\in I}$ converges, then $\{a_i\}_{i\in I}$ is a Cauchy sequence.

Proof. Suppose $\lim_{i\to\infty} a_i = L$. Given $\epsilon \in \mathbb{Q}^+$, choose an integer N such that

$$|a_i - L| < \frac{\epsilon}{2} \tag{1.3.23}$$

for all $i > N$. Then for any $i, k > N$, we have

$$|a_i - a_k| = |(a_i - L) + (L - a_k)| \le |a_i - L| + |a_k - L| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \tag{1.3.24}$$

Hence $\{a_i\}_{i\in I}$ is a Cauchy sequence. **Q.E.D.**

The proposition shows that every convergent sequence in $\mathbb{Q}$ is a Cauchy sequence, but, as the next example shows, the converse does not hold.

Example 1.3.4. Let

$$f(x) = x^2 - 2$$

and consider the sequence constructed as follows: Begin by setting $a_1 = 1$, $b_1 = 2$, and $x_1 = \frac{3}{2}$. If $f(a_1)f(x_1) < 0$, set

$$x_2 = \frac{a_1 + x_1}{2},$$

$a_2 = a_1$, and $b_2 = x_1$; otherwise, set

$$x_2 = \frac{x_1 + b_1}{2},$$

$a_2 = x_1$, and $b_2 = b_1$. In general, given a_n, x_n, and b_n, if $f(a_n)f(x_n) < 0$, set

$$x_{n+1} = \frac{a_n + x_n}{2},$$

$a_{n+1} = a_n$, and $b_{n+1} = x_n$; otherwise, set

$$x_{n+1} = \frac{x_n + b_n}{2},$$

$a_{n+1} = x_n$, and $b_{n+1} = b_n$. Note that for any positive integer N, $f(a_N) < 0$, $f(b_N) > 0$, and

$$a_N < x_i < b_N$$

for all $i > N$. Moreover,

$$|b_N - a_N| = \frac{1}{2^{N-1}},$$

so

$$|x_i - x_k| < \frac{1}{2^{N-1}}$$

for all $i, k > N$. Hence given any $\epsilon \in \mathbb{Q}^+$, if we choose an integer N such that $2^{N-1} > \frac{1}{\epsilon}$, then

$$|x_i - x_k| < \frac{1}{2^{N-1}} < \epsilon$$

for all $i, k > N$, showing that $\{x_i\}_{i=1}^{\infty}$ is a Cauchy sequence. Now suppose $\{x_i\}_{i=1}^{\infty}$ converges to $s \in \mathbb{Q}$. Note that we must have

$$a_i \le s \le b_i$$

for all $i \in \mathbb{Z}^+$. If $f(s) < 0$, then, since the set $\{a : a \in \mathbb{Q}^+, a^2 < 2\}$ does not have a supremum, there exists $t \in \mathbb{Q}^+$ such that $s < t$ and $f(t) < 0$. If we choose N so that

$$\frac{1}{2^{N-1}} < t - s,$$

then

$$|s - b_N| \le |a_N - b_N| = \frac{1}{2^{N-1}} < t - s.$$

Hence $b_N < t$, which implies that $f(b_N) < 0$, contradicting the construction of $\{b_i\}_{i=1}^{\infty}$. Hence we must have $f(s) > 0$. But if $f(s) > 0$, then there exists $t \in \mathbb{Q}^+$ such that $t < s$ and $f(t) > 0$. We can then choose N so that $t < a_N$, implying that $f(a_N) > 0$, contradicting the construction of $\{a_i\}_{i=1}^{\infty}$. Hence we must have $f(s) = 0$, which is not possible since $s \in \mathbb{Q}$. Thus we must conclude that $\{x_i\}_{i=1}^{\infty}$ does not converge.

1.4 Real Numbers

Let C be the set of all Cauchy sequences of rational numbers. We define a relation on C as follows: If $\{a_i\}_{i\in I}$ and $\{b_j\}_{j\in J}$ are Cauchy sequences in $\mathbb{Q}$, then $\{a_i\}_{i\in I} \sim \{b_j\}_{j\in J}$, which we will write more simply as $a_i \sim b_i$, if for every rational number $\epsilon > 0$, there exists an integer N such that

$$|a_i - b_i| < \epsilon \tag{1.4.1}$$

whenever $i > N$. This relation is clearly reflexive and symmetric. To show that it is also transitive, and hence an equivalence relation, suppose $a_i \sim b_i$ and $b_i \sim c_i$. Given $\epsilon \in \mathbb{Q}^+$, choose N so that

$$|a_i - b_i| < \frac{\epsilon}{2} \tag{1.4.2}$$

for all $i > N$ and M so that

$$|b_i - c_i| < \frac{\epsilon}{2} \tag{1.4.3}$$

for all $i > M$. Let L be the larger of N and M. Then, for all $i > L$,

$$|a_i - c_i| \leq |a_i - b_i| + |b_i - c_i| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \tag{1.4.4}$$

Hence $a_i \sim c_i$.

Definition 1.4.1. Using the equivalence relation just defined, we call the set of equivalence classes of C the *real numbers*, denoted $\mathbb{R}$.

Note that if $a \in \mathbb{Q}$, we may identify a with the equivalence class of the sequence $\{b_i\}_{i=1}^{\infty}$ where $b_i = a$, $i = 1, 2, 3, \ldots$, and thus consider $\mathbb{Q}$ to be a subset of $\mathbb{R}$.

Exercise 1.4.1. Suppose $\{a_i\}_{i\in I}$ and $\{b_i\}_{i\in J}$ are sequences in $\mathbb{Q}$ with

$$\lim_{i\to\infty} a_i = \lim_{i\to\infty} b_i.$$

Show that $a_i \sim b_i$.

1.4.1 Field properties

Suppose $\{a_i\}_{i\in I}$ and $\{b_j\}_{j\in J}$ are both Cauchy sequences of rational numbers. Let $K = I \cap J$ and define a new sequence $\{s_k\}_{k\in K}$ by setting $s_k = a_k + b_k$. Given any rational $\epsilon > 0$, choose integers N and M such that

$$|a_i - a_j| < \frac{\epsilon}{2} \tag{1.4.5}$$

for all $i, j > N$ and

$$|b_i - b_j| < \frac{\epsilon}{2} \tag{1.4.6}$$

for all $i, j > M$. If L is the larger of N and M, then, for all $i, j > L$,

$$|s_i - s_j| = |(a_i - a_j) + (b_i - b_j)| \leq |a_i - a_j| + |b_i - b_j| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \tag{1.4.7}$$

showing that $\{s_i\}_{k\in K}$ is also a Cauchy sequence. Moreover, suppose $a_i \sim c_i$ and $b_i \sim d_i$. Given $\epsilon \in \mathbb{Q}^+$, choose N so that

$$|a_i - c_i| < \frac{\epsilon}{2} \tag{1.4.8}$$

for all $i > N$ and choose M so that

$$|b_i - d_i| < \frac{\epsilon}{2} \tag{1.4.9}$$

for all $i > M$. If L is the larger of N and M, then, for all $i > L$,

$$|(a_i + b_i) - (c_i + d_i)| \leq |a_i - c_i| + |b_i - d_i| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \tag{1.4.10}$$

Hence $a_i + b_i \sim c_i + d_i$. Thus if $u, v \in \mathbb{R}$, with u being the equivalence class of $\{a_i\}_{i\in I}$ and v being the equivalence class of $\{b_j\}_{j\in J}$, then we may unambiguously define $u + v$ to be the equivalence class of $\{a_i + b_i\}_{i\in K}$, where $K = I \cap J$.

Suppose $\{a_i\}_{i\in I}$ and $\{b_j\}_{j\in J}$ are both Cauchy sequences of rational numbers. Let $K = I \cap J$ and define a new sequence $\{p_k\}_{k\in K}$ by setting $p_k = a_k b_k$. Let $B > 0$ be an upper bound for the set $\{|a_i| : i \in I\} \cup \{|b_j| : j \in J\}$. Given $\epsilon > 0$, choose integers N and M such that

$$|a_i - a_j| < \frac{\epsilon}{2B} \tag{1.4.11}$$

for all $i, j > N$ and

$$|b_i - b_j| < \frac{\epsilon}{2B} \tag{1.4.12}$$

for all $i, j > M$. If L is the larger of N and M, then, for all $i, j > L$,

$$\begin{aligned} |p_i - p_j| &= |a_i b_i - a_j b_j| \\ &= |a_i b_i - a_j b_i + a_j b_i - a_j b_j| \\ &= |b_i(a_i - a_j) + a_j(b_i - b_j| \\ &\leq |b_i(a_i - a_j)| + |a_j(b_i - b_j)| \\ &= |b_i||a_i - a_j| + |a_j||b_i - b_j| \\ &< B\frac{\epsilon}{2B} + B\frac{\epsilon}{2B} \\ &= \epsilon. \end{aligned} \tag{1.4.13}$$

Hence $\{p_k\}_{k\in K}$ is a Cauchy sequence.

Now suppose $\{c_i\}_{i\in H}$ and $\{d_i\}_{i\in G}$ are Cauchy sequences with $a_i \sim c_i$ and $b_i \sim d_i$. Let $B > 0$ be an upper bound for the set $\{|b_j| : j \in J\} \cup \{|c_i| : i \in H\}$. Given $\epsilon > 0$, choose integers N and M such that

$$|a_i - c_i| < \frac{\epsilon}{2B} \tag{1.4.14}$$

for all $i > N$ and

$$|b_i - d_i| < \frac{\epsilon}{2B} \tag{1.4.15}$$

for all $i > M$. If L is the larger of N and M, then, for all $i > L$,

$$\begin{aligned} |a_ib_i - c_id_i| &= |a_ib_i - b_ic_i + b_ic_i - c_id_i| \\ &= |b_i(a_i - c_i) + c_i(b_i - d_i| \\ &\le |b_i(a_i - c_i)| + |c_i(b_i - d_i)| \\ &= |b_i||a_i - c_i| + |c_i||b_i - d_i| \\ &< B\frac{\epsilon}{2B} + B\frac{\epsilon}{2B} \\ &= \epsilon. \end{aligned} \tag{1.4.16}$$

Hence $a_ib_i \sim c_id_i$. Thus if $u, v \in \mathbb{R}$, with u being the equivalence class of $\{a_i\}_{i\in I}$ and v being the equivalence class of $\{b_j\}_{j\in J}$, then we may unambiguously define uv to be the equivalence class of $\{a_ib_i\}_{i\in K}$, where $K = I \cap J$.

If $u \in \mathbb{R}$, we define $-u = (-1)u$. Note that if $\{a_i\}_{i\in I}$ is a Cauchy sequence of rational numbers in the equivalence class of u, then $\{-a_i\}_{i\in I}$ is a Cauchy sequence in the equivalence class of $-u$.

We will say that a sequence $\{a_i\}_{i\in I}$ is *bounded away from* 0 if there exists a rational number $\delta > 0$ and an integer N such that $|a_i| > \delta$ for all $i > N$. It should be clear that any sequence which converges to 0 is not bounded away from 0. Moreover, as a consequence of the next exercise, any Cauchy sequence which does not converge to 0 must be bounded away from 0.

Exercise 1.4.2. Suppose $\{a_i\}_{i\in I}$ is a Cauchy sequence which is not bounded away from 0. Show that the sequence converges and $\lim_{i\to\infty} a_i = 0$.

Exercise 1.4.3. Suppose $\{a_i\}_{i\in I}$ is a Cauchy sequence which is bounded away from 0 and $a_i \sim b_i$. Show that $\{b_j\}_{j\in J}$ is also bounded away from 0.

Now suppose $\{a_i\}_{i\in I}$ is a Cauchy sequence which is bounded away from 0 and choose $\delta > 0$ and N so that $|a_i| > \delta$ for all $i > N$. Define a new sequence $\{c_i\}_{i=N+1}^{\infty}$ by setting

$$c_i = \frac{1}{a_i}, i = N+1, N+2, \ldots. \tag{1.4.17}$$

Given $\epsilon > 0$, choose M so that

$$|a_i - a_j| < \epsilon\delta^2 \tag{1.4.18}$$

for all $i, j > M$. Let L be the larger of N and M. Then, for all $i, j > L$, we have

$$\begin{aligned} |c_i - c_j| &= \left|\frac{1}{a_i} - \frac{1}{a_j}\right| \\ &= \left|\frac{a_j - a_i}{a_i a_j}\right| \\ &= \frac{|a_j - a_i|}{|a_i a_j|} \\ &< \frac{\epsilon\delta^2}{\delta^2} \\ &= \epsilon. \end{aligned} \tag{1.4.19}$$

Hence $\{c_i\}_{i=N+1}^{\infty}$ is a Cauchy sequence.

Now suppose $\{b_j\}_{j\in J}$ is a Cauchy sequence with $a_i \sim b_i$. By Exercise 1.4.3 we know that $\{b_j\}_{j\in J}$ is also bounded away from 0, so choose $\gamma > 0$ and K such that $|b_j| > \gamma$ for all $j > K$. Given $\epsilon > 0$, choose P so that

$$|a_i - b_i| < \epsilon\delta\gamma. \tag{1.4.20}$$

for all $i > P$. Let S be the larger of N, K, and P. Then, for all $i, j > S$, we have

$$\begin{aligned} \left|\frac{1}{a_i} - \frac{1}{b_i}\right| &= \left|\frac{b_i - a_i}{a_i b_i}\right| \\ &= \frac{|b_i - a_i|}{|a_i b_i|} \\ &< \frac{\epsilon\delta\gamma}{\delta\gamma} \\ &= \epsilon. \end{aligned} \tag{1.4.21}$$

Hence $\frac{1}{a_i} \sim \frac{1}{b_i}$. Thus if $u \neq 0$ is a real number which is the equivalence class of $\{a_i\}_{i\in I}$ (necessarily bounded away from 0), then we may define

$$a^{-1} = \frac{1}{a} \tag{1.4.22}$$

to be the equivalence class of

$$\left\{\frac{1}{a_i}\right\}_{i=N+1}^{\infty}, \tag{1.4.23}$$

where N has been chosen so that $|a_i| > \delta$ for all $i > N$ and some $\delta > 0$.

It follows immediately from these definitions that $\mathbb{R}$ is a field. That is:

1. $a + b = b + a$ for all $a, b \in \mathbb{R}$;

2. $(a + b) + c = a + (b + c)$ for all $a, b, c \in \mathbb{R}$;

3. $ab = ba$ for all $a, b \in \mathbb{R}$;

4. $(ab)c = a(bc)$ for all $a, b, c \in \mathbb{R}$;

5. $a(b + c) = ab + ac$ for all $a, b, c \in \mathbb{R}$;

6. $a + 0 = a$ for all $a \in \mathbb{R}$;

7. $a + (-a) = 0$ for all $a \in \mathbb{R}$;

8. $1a = a$ for all $a \in \mathbb{R}$;

9. if $a \in \mathbb{R}$, $a \neq 0$, then $aa^{-1} = 1$.

1.4.2 Order and metric properties

Definition 1.4.2. Given $u \in \mathbb{R}$, we say that u is *positive*, written $u > 0$, if u is the equivalence class of a Cauchy sequence $\{a_i\}_{i \in I}$ for which there exists a rational number $\epsilon > 0$ and an integer N such that $a_i > \epsilon$ for every $i > N$. A real number $u \in \mathbb{R}$ is said to be *negative* if $-u > 0$. We let $\mathbb{R}^+$ denote the set of all positive real numbers.

Exercise 1.4.4. Show that if $u \in \mathbb{R}$, then one and only one of the following is true: (a) $u > 0$, (b) $u < 0$, or (c) $u = 0$.

Exercise 1.4.5. Show that if $a, b \in \mathbb{R}^+$, then $a + b \in \mathbb{R}^+$.

Definition 1.4.3. Given real numbers u and v, we say u is *greater* than v, written $u > v$, or, equivalently, v is *less* than u, written, $v < u$, if $u - v > 0$. We write $u \geq v$, or, equivalently, $v \leq u$, to indicate that u is either greater than or equal to v. We say that u is *nonnegative* if $u \geq 0$.

Exercise 1.4.6. Show that $\mathbb{R}$ is an ordered field, that is, verify the following:

a. For any $a, b \in \mathbb{R}$, one and only one of the following must hold: (i) $a < b$, (ii) $a = b$, (iii) $a > b$.

b. If $a, b, c \in \mathbb{R}$ with $a < b$ and $b < c$, then $a < c$.

c. If $a, b, c \in \mathbb{R}$ with $a < b$, then $a + c < b + c$.

d. If $a, b \in \mathbb{R}$ with $a > 0$ and $b > 0$, then $ab > 0$.

Exercise 1.4.7. Show that if $a, b \in \mathbb{R}$ with $a > 0$ and $b < 0$, then $ab < 0$.

Exercise 1.4.8. Show that if $a, b, c \in \mathbb{R}$ with $a < b$, then $ac < bc$ if $c > 0$ and $ac > bc$ if $c < 0$.

Exercise 1.4.9. Show that if $a, b \in \mathbb{R}$ with $a < b$, then for any real number λ with $0 < \lambda < 1$, $a < \lambda a + (1 - \lambda)b < b$.

Definition 1.4.4. For any $a \in \mathbb{R}$, we call

$$|a| = \begin{cases} a, & \text{if } a \geq 0, \\ -a, & \text{if } a < 0, \end{cases} \tag{1.4.24}$$

the *absolute value* of a.

Exercise 1.4.10. Show that for any $a \in \mathbb{R}$, $-|a| \leq a \leq |a|$.

Proposition 1.4.1. For any $a, b \in \mathbb{R}$, $|a + b| \leq |a| + |b|$.

Proof. If $a + b \geq 0$, then

$$|a| + |b| - |a + b| = |a| + |b| - a - b = (|a| - a) + (|b| - b). \tag{1.4.25}$$

Both of the terms on the right are nonnegative by Exercise 1.4.10. Hence the sum is nonnegative and the proposition follows. If $a + b < 0$, then

$$|a| + |b| - |a + b| = |a| + |b| + a + b = (|a| + a) + (|b| + b). \tag{1.4.26}$$

Again, both of the terms on the right are nonnegative by Exercise 1.4.10. Hence the sum is nonnegative and the proposition follows. **Q.E.D.**

It is now easy to show that the absolute value function satisfies

1. $|a - b| \geq 0$ for all $a, b \in \mathbb{R}$, with $|a - b| = 0$ if and only if $a = b$,

2. $|a - b| = |b - a|$ for all $a, b \in \mathbb{R}$,

3. $|a - b| \leq |a - c| + |c - b|$ for all $a, b, c \in \mathbb{R}$.

These properties show that the function

$$d(a, b) = |a - b| \tag{1.4.27}$$

is a metric, and we will call $|a - b|$ the *distance* from a to b.

Proposition 1.4.2. Given $a \in \mathbb{R}^+$, there exist $r, s \in \mathbb{Q}$ such that $0 < r < a < s$.

Proof. Let $\{u\}_{i\in I}$ be a Cauchy sequence in the equivalence class of a. Since $a > 0$, there exists a rational $\epsilon > 0$ and an integer N such that $u_i > \epsilon$ for all $i > N$. Let $r = \frac{\epsilon}{2}$. Then $u_i - r > \frac{\epsilon}{2}$ for every $i > N$, so $a - r > 0$, that is, $0 < r < a$.

Now choose an integer M so that $|u_i - u_j| < 1$ for all $i, j > M$. Let $s = u_{M+1} + 2$. Then

$$s - u_i = u_{M+1} + 2 - u_i > 1 \tag{1.4.28}$$

for all $i > M$. Hence $s > a$. **Q.E.D.**

Proposition 1.4.3. $\mathbb{R}$ is an archimedean ordered field.

Proof. Given real numbers a and b with $0 < a < b$, let r and s be rational numbers for which $0 < r < a < b < s$. Since $\mathbb{Q}$ is a an archimedean field, there exists an integer n such that $nr > s$. Hence

$$na > nr > s > b. \tag{1.4.29}$$

Q.E.D.

Proposition 1.4.4. Given $a, b \in \mathbb{R}$ with $a < b$, there exists $r \in \mathbb{Q}$ such that $a < r < b$.

Proof. Let $\{u\}_{i\in I}$ be a Cauchy sequence in the equivalence class of a and let $\{v\}_{j\in J}$ be in the equivalence class of b. Since $b - a > 0$, there exists a rational $\epsilon > 0$ and an integer N such that $v_i - u_i > \epsilon$ for all $i > N$. Now choose an integer M so that $|u_i - u_j| < \frac{\epsilon}{4}$ for all $i, j > M$. Let $r = u_{M+1} + \frac{\epsilon}{2}$. Then

$$\begin{aligned} r - u_i &= u_{M+1} + \frac{\epsilon}{2} - u_i \\ &= \frac{\epsilon}{2} - (u_i - u_{M+1}) \\ &> \frac{\epsilon}{2} - \frac{\epsilon}{4} \\ &= \frac{\epsilon}{4} \end{aligned} \tag{1.4.30}$$

for all $i > M$ and

$$\begin{aligned} v_i - r &= v_i - u_{M+1} - \frac{\epsilon}{2} \\ &= (v_i - u_i) - (u_{M+1} - u_i) - \frac{\epsilon}{2} \\ &> \epsilon - \frac{\epsilon}{4} - \frac{\epsilon}{2} \\ &= \frac{\epsilon}{4} \end{aligned} \tag{1.4.31}$$

for all i larger than the larger of N and M. Hence $a < r < b$. **Q.E.D.**

1.4.3 Upper and lower bounds

Definition 1.4.5. Let $A \subset \mathbb{R}$. If $s \in \mathbb{R}$ is such that $s \geq a$ for every $a \in A$, then we call s an *upper bound* for A. If s is an upper bound for A with the property that $s \leq t$ whenever t is an upper bound for A, then we call s the *supremum*, or *least upper bound*, of A, denoted $s = \sup A$. Similarly, if $r \in \mathbb{R}$ is such that $r \leq a$ for every $a \in A$, then we call r a *lower bound* for A. If r is a lower bound for A with the property that $r \geq t$ whenever t is a lower bound for A, then we call r the *infimum*, or *greatest lower bound*, of A, denoted $r = \inf A$.

Theorem 1.4.5. Suppose $A \subset \mathbb{R}$, $A \neq \emptyset$, has an upper bound. Then $\sup A$ exists.

Proof. Let $a \in A$ and let b be an upper bound for A. Define sequences $\{a_i\}_{i=1}^{\infty}$ and $\{b_i\}_{i=1}^{\infty}$ as follows: Let $a_1 = a$ and $b_1 = b$. For $i > 1$, let

$$c = \frac{a_{i-1} + b_{i-1}}{2}. \tag{1.4.32}$$

If c is an upper bound for A, let $a_i = a_{i-1}$ and let $b_i = c$; otherwise, let $a_i = c$ and $b_i = b_{i-1}$. Then

$$|b_i - a_i| = \frac{|b-a|}{2^{i-1}} \tag{1.4.33}$$

for $i = 1, 2, 3, \ldots$. Now, for $i = 1, 2, 3, \ldots$, let r_i be a rational number such that $a_i < r_i < b_i$. Given any $\epsilon > 0$, we may choose N so that

$$2^N > \frac{|b-a|}{\epsilon}. \tag{1.4.34}$$

Then, whenever $i > N$ and $j > N$,

$$|r_i - r_j| < |b_{N+1} - a_{N+1}| = \frac{|b-a|}{2^N} < \epsilon. \tag{1.4.35}$$

Hence $\{r_i\}_{i=1}^{\infty}$ is a Cauchy sequence. Let $s \in \mathbb{R}$ be the equivalence class of $\{r_i\}_{i=1}^{\infty}$. Note that, for $i = 1, 2, 3, \ldots$, $a_i \leq s \leq b_i$.

Now if s is not an upper bound for A, then there exists $a \in A$ with $a > s$. Let $\delta = a - s$ and choose an integer N such that

$$2^N > \frac{|b-a|}{\delta}. \tag{1.4.36}$$

Then

$$b_{N+1} \leq s + \frac{|b-a|}{2^N} < s + \delta = a. \tag{1.4.37}$$

But, by construction, b_{N+1} is an upper bound for A. Thus s must be an upper bound for A.

Now suppose t is another upper bound for A and $t < s$. Let $\delta = s - t$ and choose an integer N such that

$$2^N > \frac{|b-a|}{\delta}. \tag{1.4.38}$$

Then

$$a_{N+1} \geq s - \frac{|b-a|}{2^N} > s - \delta = t, \tag{1.4.39}$$

which implies that a_{N+1} is an upper bound for A. But, by construction, a_{N+1} is not an upper bound for A. Hence s must be the least upper bound for A, that is, $s = \sup A$. **Q.E.D.**

Exercise 1.4.11. Show that if $A \subset \mathbb{R}$ is nonempty and has a lower bound, then $\inf A$ exists. (Hint: You may wish to first show that $\inf A = -\sup(-A)$, where $-A = \{x : -x \in A\}$).

Chapter 2

Sequences and Series

2.1 Sequences

Definition 2.1.1. Let $\{a_i\}_{i \in I}$ be a sequence of real numbers. We say $\{a_i\}_{i \in I}$ *converges*, and has *limit* L, if for every real number $\epsilon > 0$ there exists an integer N such that

$$|a_i - L| < \epsilon \text{ whenever } i > N. \tag{2.1.1}$$

We say a sequence $\{a_i\}_{i \in I}$ which does not converge *diverges*.

Definition 2.1.2. We say a sequence $\{a_i\}_{i \in I}$ is *nondecreasing* if $a_{i+1} \geq a_i$ for every $i \in I$ and *increasing* if $a_{i+1} > a_i$ for every $i \in I$. We say a sequence $\{a_i\}_{i \in I}$ is *nonincreasing* if $a_{i+1} \leq a_i$ for every $i \in I$ and *decreasing* if $a_{i+1} < a_i$ for every $i \in I$.

Definition 2.1.3. We say a set $A \subset \mathbb{R}$ is *bounded* if there exists a real number M such that $|a| \leq M$ for every $a \in A$. We say a sequence $\{a_i\}_{i \in I}$ of real numbers is *bounded* if there exists a real number M such that $|a_i| \leq M$ for all $i \in I$.

Theorem 2.1.1. If $\{a_i\}_{i \in I}$ is a nondecreasing, bounded sequence of real numbers, then $\{a_i\}_{i \in I}$ converges.

Proof. Since $\{a_i\}_{i \in I}$ is bounded, the set of $A = \{a_i : i \in I\}$ has a supremum. Let $L = \sup A$. For any $\epsilon > 0$, there must exist $N \in I$ such that $a_N > L - \epsilon$ (or else $L - \epsilon$ would be an upper bound for A which is smaller than L). But then

$$L - \epsilon < a_N \leq a_i \leq L < L + \epsilon \tag{2.1.2}$$

for all $i \geq N$, that is,

$$|a_i - L| < \epsilon \tag{2.1.3}$$

for all $i \geq N$. Thus $\{a_i\}_{i \in I}$ converges and

$$L = \lim_{i \to \infty} a_i. \tag{2.1.4}$$

Q.E.D.

We conclude from the previous theorem that every nondecreasing sequence of real numbers either has a limit or is not bounded, that is, is *unbounded.*

Exercise 2.1.1. Show that a nonincreasing, bounded sequence of real numbers must converge.

Definition 2.1.4. Let $\{a_i\}_{i\in I}$ be a sequence of real numbers. If for every real number M there exists an integer N such that $a_i > M$ whenever $i > N$, then we say the sequence $\{a_i\}_{i\in I}$ *diverges to positive infinity*, denoted by

$$\lim_{i\to\infty} a_i = +\infty. \tag{2.1.5}$$

Similarly, if for every real number M there exists an integer N such that $a_i < M$ whenever $i > N$, then we say the sequence $\{a_i\}_{i\in I}$ *diverges to negative infinity*, denoted by

$$\lim_{i\to\infty} a_i = -\infty. \tag{2.1.6}$$

Exercise 2.1.2. Show that a nondecreasing sequence of real numbers either converges or diverges to positive infinity.

Exercise 2.1.3. Show that a nonincreasing sequence of real numbers either converges or diverges to negative infinity.

2.1.1 Extended real numbers

It is convenient to add the symbols $+\infty$ and $-\infty$ to the real numbers $\mathbb{R}$. Although neither $+\infty$ nor $-\infty$ is a real number, we agree to the following operational conventions:

1. Given any real number r, $-\infty < r < \infty$.

2. For any real number r,

$$r + (+\infty) = r - (-\infty) = r + \infty = +\infty, \tag{2.1.7}$$

$$r + (-\infty) = r - (+\infty) = r - \infty = -\infty, \tag{2.1.8}$$

   and

$$\frac{r}{+\infty} = \frac{r}{-\infty} = 0. \tag{2.1.9}$$

3. For any real number $r > 0$, $r \cdot (+\infty) = +\infty$ and $r \cdot (-\infty) = -\infty$.

4. For any real number $r < 0$, $r \cdot (+\infty) = -\infty$ and $r \cdot (-\infty) = +\infty$.

5. If $a_i = -\infty$, $i = 1, 2, 3, \ldots$, then $\lim_{i\to\infty} a_i = -\infty$.

6. If $a_i = +\infty$, $i = 1, 2, 3, \ldots$, then $\lim_{i\to\infty} a_i = +\infty$.

Note that with the order relation defined in this manner, $+\infty$ is an upper bound and $-\infty$ is a lower bound for any set $A \subset \mathbb{R}$. Thus if $A \subset \mathbb{R}$ does not have a finite upper bound, then $\sup A = +\infty$; similarly, if $A \subset \mathbb{R}$ does not have a finite lower bound, then $\inf A = -\infty$.

When working with extended real numbers, we refer to the elements of $\mathbb{R}$ as *finite* real numbers and the elements $+\infty$ and $-\infty$ as *infinite* real numbers.

Exercise 2.1.4. Do the extended real numbers form a field?

2.1.2 Limit superior and inferior

Definition 2.1.5. Let $\{a_i\}_{i\in I}$ be a sequence of real numbers and, for each $i \in I$, let $u_i = \sup\{a_j : j \geq i\}$. If $u_i = +\infty$ for every $i \in I$, we let

$$\limsup_{i\to\infty} a_i = +\infty; \tag{2.1.10}$$

otherwise, we let

$$\limsup_{i\to\infty} a_i = \inf\{u_i : i \in I\}. \tag{2.1.11}$$

In either case, we call $\limsup_{n\to\infty} a_n$ the *limit superior* of the sequence $\{a_i\}_{i\in I}$.

Definition 2.1.6. Let $\{a_i\}_{i\in I}$ be a sequence of real numbers and, for each $i \in I$, let $l_i = \inf\{a_j : j \geq i\}$. If $l_i = -\infty$ for every $i \in I$, we let

$$\liminf_{i\to\infty} a_i = -\infty; \tag{2.1.12}$$

otherwise, we let

$$\liminf_{i\to\infty} a_i = \sup\{l_i : i \in I\}. \tag{2.1.13}$$

In either case, we call $\liminf_{n\to\infty} a_n$ the *limit inferior* of the sequence $\{a_i\}_{i\in I}$.

Exercise 2.1.5. Given a sequence $\{a_i\}_{i\in I}$, define $\{u_i\}_{i\in I}$ and $\{l_i\}_{i\in I}$ as in the previous two definitions. Show that

$$\limsup_{i\to\infty} a_i = \lim_{i\to\infty} u_i \tag{2.1.14}$$

and

$$\liminf_{i\to\infty} a_i = \lim_{i\to\infty} l_i. \tag{2.1.15}$$

Exercise 2.1.6. Find $\limsup_{i\to\infty} a_i$ and $\liminf_{i\to\infty} a_i$ for the sequences $\{a_i\}_{i=1}^{\infty}$ as defined below.

a. $a_i = (-1)^i$

b. $a_i = i$

c. $a_i = 2^{-i}$

d. $a_i = \frac{1}{i}$

The following proposition is often called the *squeeze theorem*.

Proposition 2.1.2. Suppose $\{a_i\}_{i\in I}$, $\{b_j\}_{j\in J}$, and $\{c_k\}_{k\in K}$ are sequences of real numbers for which there exists an integer N such that $a_i \leq c_i \leq b_i$ whenever $i > N$. If

$$\lim_{i\to\infty} a_i = \lim_{i\to\infty} b_i, \tag{2.1.16}$$

then

$$\lim_{i\to\infty} c_i = \lim_{i\to\infty} a_i = \lim_{i\to\infty} b_i. \tag{2.1.17}$$

Proof. Let $L = \lim_{i\to\infty} a_i = \lim_{i\to\infty} b_i$. Suppose L is finite. Given $\epsilon > 0$, there exists an integer M such that

$$|a_i - L| < \frac{\epsilon}{3} \tag{2.1.18}$$

and

$$|b_i - L| < \frac{\epsilon}{3} \tag{2.1.19}$$

whenever $i > M$. Then

$$|a_i - b_i| \leq |a_i - L| + |L - b_i| < \frac{\epsilon}{3} + \frac{\epsilon}{3} = \frac{2\epsilon}{3} \tag{2.1.20}$$

whenever $i > M$. Let K be the larger of N and M. Then

$$|c_i - L| \leq |c_i - b_i| + |b_i - L| \leq |a_i - b_i| + |b_i - L| < \frac{2\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \tag{2.1.21}$$

whenever $i > K$. Thus $\lim_{i\to\infty} c_i = L$. The result when L is infinite is a consequence of the next two exercises. **Q.E.D.**

Exercise 2.1.7. Suppose $\{a_i\}_{i\in I}$ and $\{c_k\}_{k\in K}$ are sequences for which there exists an integer N such that $a_i \leq c_i$ whenever $i > N$. Show that if $\lim_{i\to\infty} a_i = +\infty$, then $\lim_{i\to\infty} c_i = +\infty$.

Exercise 2.1.8. Suppose $\{b_j\}_{j\in J}$ and $\{c_k\}_{k\in K}$ are sequences for which there exists an integer N such that $c_i \leq b_i$ whenever $i > N$. Show that if $\lim_{i\to\infty} b_i = -\infty$, then $\lim_{i\to\infty} c_i = -\infty$.

Exercise 2.1.9. Suppose $\{a_i\}_{i\in I}$ and $\{b_j\}_{j\in J}$ are sequences of real numbers with $a_i \leq b_i$ for all i larger than some integer N. Assuming both sequences converge, show that

$$\lim_{i\to\infty} a_i \leq \lim_{i\to\infty} b_i. \tag{2.1.22}$$

Exercise 2.1.10. Show that for any sequence $\{a_i\}_{i\in I}$,

$$\liminf_{i\to\infty} a_i \leq \limsup_{i\to\infty} a_i. \tag{2.1.23}$$

Proposition 2.1.3. Suppose $\{a_i\}_{i\in I}$ is a sequence for which

$$\limsup_{i\to\infty} a_i = \liminf_{i\to\infty} a_i. \tag{2.1.24}$$

Then

$$\lim_{i\to\infty} a_i = \limsup_{i\to\infty} a_i = \liminf_{i\to\infty} a_i. \tag{2.1.25}$$

Proof. Let $u_i = \sup\{a_k : k \geq i\}$ and $l_i = \inf\{a_k : k \geq i\}$. Then $l_i \leq a_i \leq u_i$ for all $i \in I$. Now

$$\lim_{i\to\infty} l_i = \liminf_{i\to\infty} a_i = \limsup_{i\to\infty} a_i = \lim_{i\to\infty} u_i, \tag{2.1.26}$$

so the result follows from the squeeze theorem. **Q.E.D.**

Exercise 2.1.11. Suppose u is a real number such that $u \geq 0$ and $u < \epsilon$ for any real number $\epsilon > 0$. Show that $u = 0$.

2.1.3 Completeness

Definition 2.1.7. Suppose $\{a_i\}_{i\in I}$ is a sequence in $\mathbb{R}$. We call $\{a_i\}_{i\in I}$ a *Cauchy sequence* if for every $\epsilon > 0$ there exists an integer N such that

$$|a_i - a_j| < \epsilon \tag{2.1.27}$$

whenever both $i > N$ and $j > N$.

Theorem 2.1.4. Suppose $\{a_i\}_{i\in I}$ is a Cauchy sequence in $\mathbb{R}$. Then $\{a_i\}_{i\in I}$ converges to a limit $L \in \mathbb{R}$.

Proof. Let $u_i = \sup\{a_k : k \geq i\}$ and $l_i = \inf\{a_k : k \geq i\}$. Given any $\epsilon > 0$, there exists $N \in \mathbb{Z}$ such that $|a_i - a_j| < \epsilon$ for all $i, j > N$. Thus, for all $i, j > N$, $a_i < a_j + \epsilon$, and so

$$a_i \leq \inf\{a_j + \epsilon : j \geq i\} = l_i + \epsilon \tag{2.1.28}$$

for all $i > N$. Since $\{l_i\}_{i \in I}$ is a nondecreasing sequence,

$$a_i \leq \sup\{l_i + \epsilon : i \in I\} = \liminf_{i \to \infty} a_i + \epsilon \tag{2.1.29}$$

for all $i > N$. Hence

$$u_i = \sup\{a_k : k \geq i\} \leq \liminf_{i \to \infty} a_i + \epsilon \tag{2.1.30}$$

for all $i > N$. Thus

$$\limsup_{i \to \infty} a_i = \inf\{u_i : i \in I\} \leq \liminf_{i \to \infty} a_i + \epsilon. \tag{2.1.31}$$

Since $\liminf_{i \to \infty} a_i \leq \limsup_{i \to \infty} a_i$, it follows that

$$|\limsup_{i \to \infty} a_i - \liminf_{i \to \infty} a_i| \leq \epsilon. \tag{2.1.32}$$

Since this is true for every $\epsilon > 0$, we have $\limsup_{i \to \infty} a_i = \liminf_{i \to \infty} a_i$, and so $\{a_i\}_{i \in I}$ converges by Proposition 2.1.3. **Q.E.D.**

As a consequence of the previous theorem, we say that $\mathbb{R}$ is a *complete* metric space.

Exercise 2.1.12. Suppose $A \subset \mathbb{R}$, $A \neq \emptyset$, and $s = \sup A$. Show that there exists a sequence $\{a_i\}_{i=1}^{\infty}$ with $a_i \in A$ such that $\lim_{i \to \infty} a_i = s$.

Exercise 2.1.13. Given a real number $x \geq 0$, show that there exists a real number $s \geq 0$ such that $s^2 = x$.

We let $\sqrt{x}$ denote the number s in the previous exercise, the *square root* of x.

2.1.4 Some basic results about sequences

Proposition 2.1.5. Suppose $\{x_i\}_{i \in I}$ is a convergent sequence in $\mathbb{R}$, α is a real number, and $L = \lim_{i \to \infty} x_i$. Then the sequence $\{\alpha x_i\}_{i \in I}$ converges and

$$\lim_{i \to \infty} \alpha x_i = \alpha L. \tag{2.1.33}$$

Proof. If $\alpha = 0$, then $\{\alpha x_i\}_{i \in I}$ clearly converges to 0. So assume $\alpha \neq 0$. Given $\epsilon > 0$, choose an integer N such that

$$|x_i - L| < \frac{\epsilon}{|\alpha|} \tag{2.1.34}$$

whenever $i > N$. Then for any $i > N$ we have

$$|\alpha x_i - \alpha L| = |\alpha||x_i - L| < |\alpha|\frac{\epsilon}{|\alpha|} = \epsilon. \tag{2.1.35}$$

Thus $\lim_{i \to \infty} \alpha x_i = \alpha L$. **Q.E.D.**

Proposition 2.1.6. Suppose $\{x_i\}_{i\in I}$ and $\{y_i\}_{i\in I}$ are convergent sequences in $\mathbb{R}$ with $L = \lim_{i\to\infty} x_i$ and $M = \lim_{i\to\infty} y_i$. Then the sequence $\{x_i + y_i\}_{i\in I}$ converges and

$$\lim_{i\to\infty}(x_i + y_i) = L + M. \tag{2.1.36}$$

Exercise 2.1.14. Prove the previous proposition.

Proposition 2.1.7. Suppose $\{x_i\}_{i\in I}$ and $\{y_i\}_{i\in I}$ are convergent sequences in $\mathbb{R}$ with $L = \lim_{i\to\infty} x_i$ and $M = \lim_{i\to\infty} y_i$. Then the sequence $\{x_i y_i\}_{i\in I}$ converges and

$$\lim_{i\to\infty} x_i y_i = LM. \tag{2.1.37}$$

Exercise 2.1.15. Prove the previous proposition.

Proposition 2.1.8. Suppose $\{x_i\}_{i\in I}$ and $\{y_i\}_{i\in I}$ are convergent sequences in $\mathbb{R}$ with $L = \lim_{i\to\infty} x_i$, $M = \lim_{i\to\infty} y_i$, and $y_i \neq 0$ for all $i \in I$. If $M \neq 0$, then the sequence $\left\{\frac{x_i}{y_i}\right\}_{i\in I}$ converges and

$$\lim_{i\to\infty} \frac{x_i}{y_i} = \frac{L}{M}. \tag{2.1.38}$$

Proof. Since $M \neq 0$ and $M = \lim_{i\to\infty} y_i$, we may choose an integer N such that

$$|y_i| > \frac{|M|}{2} \tag{2.1.39}$$

whenever $i > N$. Let B be an upper bound for $\{|x_i| : i \in I\} \cup \{|y_i| : i \in I\}$. Given any $\epsilon > 0$, we may choose an integer P such that

$$|x_i - L| < \frac{M^2\epsilon}{4B} \tag{2.1.40}$$

and

$$|y_i - M| < \frac{M^2\epsilon}{4B} \tag{2.1.41}$$

whenever $i > P$. Let K be the larger of N and P. Then, for any $i > K$, we have

$$\begin{aligned}\left|\frac{x_i}{y_i}-\frac{L}{M}\right| &= \frac{|x_iM-y_iL|}{|y_iM|}\\ &= \frac{|x_iM-x_iy_i+x_iy_i-y_iL|}{|y_iM|}\\ &\leq \frac{|x_i||M-y_i|+|y_i||x_i-L|}{|y_iM|}\\ &< \frac{B\dfrac{M^2\epsilon}{4B}+B\dfrac{M^2\epsilon}{4B}}{\dfrac{M^2}{2}}\\ &= \epsilon. \end{aligned} \tag{2.1.42}$$

Thus

$$\lim_{i\to\infty}\frac{x_i}{y_i}=\frac{L}{M}. \tag{2.1.43}$$

Q.E.D.

Exercise 2.1.16. a. Show that $\lim_{n\to\infty}\frac{1}{n}=0$.

b. Show that

$$\lim_{n\to\infty}\frac{1}{n^2}=0$$

by (i) using the definition of limit directly and then (ii) using previous results.

Exercise 2.1.17. Show that for any positive integer k,

$$\lim_{n\to\infty}\frac{1}{n^k}=0.$$

Example 2.1.1. We may combine the properties of this section to compute

$$\begin{aligned}\lim_{n\to\infty}\frac{5n^3+3n-6}{2n^3+2n^2-7} &= \lim_{n\to\infty}\frac{5+\dfrac{3}{n^2}-\dfrac{6}{n^3}}{2+\dfrac{2}{n}-\dfrac{7}{n^3}}\\ &= \frac{\lim_{n\to\infty}5+3\lim_{n\to\infty}\dfrac{1}{n^2}-6\lim_{n\to\infty}\dfrac{1}{n^3}}{\lim_{n\to\infty}2+2\lim_{n\to\infty}\dfrac{1}{n}-7\lim_{n\to\infty}\dfrac{1}{n^3}}\\ &= \frac{5+0+0}{2+0+0}\\ &= \frac{5}{2}.\end{aligned}$$

Exercise 2.1.18. Evaluate

$$\lim_{n\to\infty} \frac{3n^5 + 8n^3 - 6n}{8n^5 + 2n^4 - 31},$$

carefully showing each step.

Proposition 2.1.9. Suppose $\{x_i\}_{i\in I}$ is a convergent sequence of nonnegative real numbers with $L = \lim_{i\to\infty} x_i$. Then the sequence $\{\sqrt{x_i}\}_{i\in I}$ converges and

$$\lim_{i\to\infty} \sqrt{x_i} = \sqrt{L}. \tag{2.1.44}$$

Proof. Let $\epsilon > 0$ be given. Suppose $L > 0$ and note that

$$|x_i - L| = \left|\sqrt{x_i} - \sqrt{L}\right| \left|\sqrt{x_i} + \sqrt{L}\right| \tag{2.1.45}$$

implies that

$$\left|\sqrt{x_i} - \sqrt{L}\right| = \frac{|x_i - L|}{\left|\sqrt{x_i} + \sqrt{L}\right|} \tag{2.1.46}$$

for any $i \in I$. Choose an integer N such that

$$|x_i - L| < \sqrt{L}\epsilon \tag{2.1.47}$$

whenever $i > N$. Then, for any $i > N$,

$$|\sqrt{x_i} - \sqrt{L}| = \frac{|x_i - L|}{|\sqrt{x_i} + \sqrt{L}|} < \frac{\sqrt{L}\epsilon}{\sqrt{L}} = \epsilon. \tag{2.1.48}$$

Hence $\lim_{i\to\infty} \sqrt{x_i} = \sqrt{L}$.

If $L = 0$, $\lim_{i\to\infty} x_i = 0$, so we may choose an integer N such that $|x_i| < \epsilon^2$ for all $i > N$. Then

$$|\sqrt{x_i}| < \epsilon \tag{2.1.49}$$

whenever $i > N$, so $\lim_{i\to\infty} \sqrt{x_i} = 0$. **Q.E.D.**

Exercise 2.1.19. Evaluate

$$\lim_{n\to\infty} \frac{\sqrt{3n^2 + 1}}{5n + 6},$$

carefully showing each step.

Exercise 2.1.20. Given real numbers $r > 0$ and α, show that (a) $\alpha r < r$ if $0 < \alpha < 1$ and (b) $r < \alpha r$ if $\alpha > 1$.

Proposition 2.1.10. If $x \in \mathbb{R}$ and $|x| < 1$, then

$$\lim_{n\to\infty} x^n = 0. \tag{2.1.50}$$

Proof. We first assume $x \geq 0$. Then the sequence $\{x^n\}_{n=1}^{\infty}$ is nonincreasing and bounded below by 0. Hence the sequence converges. Let $L = \lim_{n\to\infty} x^n$. Then

$$L = \lim_{n\to\infty} x^n = x \lim_{n\to\infty} x^{n-1} = xL, \tag{2.1.51}$$

from which it follows that $L(1-x) = 0$. Since $1 - x > 0$, we must have $L = 0$. The result for $x < 0$ follows from the next exercise. **Q.E.D.**

Exercise 2.1.21. Show that $\lim_{n\to\infty} |a_n| = 0$ if and only if $\lim_{n\to\infty} a_n = 0$.

2.1.5 Subsequences

Definition 2.1.8. Given a sequence $\{x_i\}_{i=m}^{\infty}$, suppose $\{n_k\}_{k=1}^{\infty}$ is an increasing sequence of integers with

$$m \leq n_1 < n_2 < n_3 < \cdots . \tag{2.1.52}$$

Then we call the sequence $\{x_{n_k}\}_{k=1}^{\infty}$ a *subsequence* of $\{x_i\}_{i=m}^{\infty}$.

Example 2.1.2. The sequence $\{x_{2k}\}_{k=1}^{\infty}$ is a subsequence of the sequence $\{x_i\}_{i=1}^{\infty}$. For example, $\left\{\frac{1}{2i}\right\}_{i=1}^{\infty}$ is a subsequence of $\left\{\frac{1}{i}\right\}_{i=1}^{\infty}$.

Exercise 2.1.22. Suppose $\{x_i\}_{i=m}^{\infty}$ converges with $\lim_{i\to\infty} x_i = L$. Show that every subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_i\}_{i=m}^{\infty}$ also converges and $\lim_{k\to\infty} x_{n_k} = L$.

Exercise 2.1.23. Suppose $\{x_i\}_{i=m}^{\infty}$ diverges to $+\infty$. Show that every subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_i\}_{i=m}^{\infty}$ also diverges to $+\infty$.

Exercise 2.1.24. Suppose $\{x_i\}_{i=m}^{\infty}$ diverges to $-\infty$. Show that every subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_i\}_{i=m}^{\infty}$ also diverges to $-\infty$.

Definition 2.1.9. Given a sequence $\{x_i\}_{i=m}^{\infty}$, we call any extended real number λ which is the limit of a subsequence of $\{x_i\}_{i=m}^{\infty}$ a *subsequential limit* of $\{x_i\}_{i=m}^{\infty}$.

Example 2.1.3. -1 and 1 are both subsequential limits of $\{(-1)^i\}_{i=0}^{\infty}$.

Exercise 2.1.25. Suppose the sequence $\{x_i\}_{i=m}^{\infty}$ is not bounded. Show that either $-\infty$ or $+\infty$ is a subsequential limit of $\{x_i\}_{i=m}^{\infty}$.

Proposition 2.1.11. Suppose Λ is the set of all subsequential limits of the sequence $\{x_i\}_{i=m}^{\infty}$. Then $\Lambda \neq \emptyset$.

Proof. By the previous exercise, the proposition is true if $\{x_i\}_{i=m}^{\infty}$ is not bounded. So suppose $\{x_i\}_{i=m}^{\infty}$ is bounded and choose real numbers a and b such that $a \leq x_i \leq b$ for all $i \geq m$. Construct sequences $\{a_i\}_{i=1}^{\infty}$ and $\{b_i\}_{i=1}^{\infty}$ as follows: Let $a_1 = a$ and $b_1 = b$. For $i \geq 1$, let

$$c = \frac{a_{i-1} + b_{i-1}}{2}. \tag{2.1.53}$$

If there exists an integer N such that $a_{i-1} \leq x_j \leq c$ for all $j > N$, let $a_i = a_{i-1}$ and $b_i = c$; otherwise, let $a_i = c$ and $b_i = b_{i-1}$. Let $n_1 = m$ and, for $k = 2, 3, 4, \ldots$, let n_k be the smallest integer for which $n_k > n_{k-1}$ and $a_k \leq x_{n_k} \leq b_k$. Then $\{x_{n_k}\}_{k=1}^{\infty}$ is a Cauchy sequence which is a subsequence of $\{x_i\}_{i=m}^{\infty}$. Thus $\{x_{n_k}\}_{k=1}^{\infty}$ converges and $\Lambda \neq \emptyset$. **Q.E.D.**

Exercise 2.1.26. Suppose $A \subset \mathbb{R}$ and $B \subset \mathbb{R}$ with $a \leq b$ for every $a \in A$ and $b \in B$. Show that $\sup A \leq \inf B$.

Proposition 2.1.12. Let Λ be the set of subsequential limits of a sequence $\{x_i\}_{i=m}^{\infty}$. Then

$$\limsup_{i \to \infty} x_i = \sup \Lambda. \tag{2.1.54}$$

Proof. Let $s = \sup \Lambda$ and, for $i \geq m$, $u_i = \sup\{x_j : j \geq i\}$. Now since $x_j \leq u_i$ for all $j \geq i$, it follows that $\lambda \leq u_i$ for every $\lambda \in \Lambda$ and $i \geq m$. Hence, from the previous exercise, $s \leq \inf\{u_i : i \geq m\} = \limsup_{i \to \infty} x_i$.

Now suppose $s < \limsup_{i \to \infty} x_i$. Then there exists a real number t such that $s < t < \limsup_{i \to \infty} x_i$. In particular, $t < u_i$ for every $i \geq m$. Let n_1 be the smallest integer for which $n_1 \geq m$ and $x_{n_1} > t$. For $k = 2, 3, 4, \ldots$, let n_k be the smallest integer for which $n_k > n_{k-1}$ and $x_{n_k} > t$. Then $\{x_{n_k}\}_{k=1}^{\infty}$ is a subsequence of $\{x_i\}_{i=m}^{\infty}$ which has a subsequential limit $\lambda \geq t$. Since λ is also then a subsequential limit of $\{x_i\}_{i=m}^{\infty}$, we have $\lambda \in \Lambda$ and $\lambda \geq t > s$, contradicting $s = \sup \Lambda$. Hence we must have $\limsup_{i \to \infty} x_i = \sup \Lambda$. **Q.E.D.**

Exercise 2.1.27. Let Λ be the set of subsequential limits of a sequence $\{x_i\}_{i=m}^{\infty}$. Show that

$$\liminf_{i \to \infty} x_i = \inf \Lambda.$$

2.2 Infinite series

Definition 2.2.1. Given a sequence $\{a_i\}_{i=m}^{\infty}$, let $\{s_n\}_{n=m}^{\infty}$ be the sequence defined by

$$s_n = \sum_{i=m}^{n} a_i. \tag{2.2.1}$$

We call the sequence $\{s_n\}_{n=m}^{\infty}$ an *infinite series*. If $\{s_n\}_{n=m}^{\infty}$ converges, we call

$$s = \lim_{n\to\infty} s_n \tag{2.2.2}$$

the *sum* of the series. For any integer $n \geq m$, we call s_n a *partial sum* of the series.

We will use the notation

$$\sum_{i=m}^{\infty} a_i \tag{2.2.3}$$

to denote either $\{s_n\}_{n=m}^{\infty}$, the infinite series, or s, the sum of the infinite series. Of course, if $\{s_n\}_{n=m}^{\infty}$ diverges, then we say $\sum_{i=m}^{\infty} a_i$ *diverges*.

Exercise 2.2.1. Suppose $\sum_{i=m}^{\infty} a_i$ converges and $\beta \in \mathbb{R}$. Show that $\sum_{i=m}^{\infty} \beta a_i$ also converges and

$$\sum_{i=m}^{\infty} \beta a_i = \beta \sum_{i=m}^{\infty} a_i.$$

Exercise 2.2.2. Suppose both $\sum_{i=m}^{\infty} a_i$ and $\sum_{i=m}^{\infty} b_i$ converge. Show that $\sum_{i=m}^{\infty}(a_i + b_i)$ converges and

$$\sum_{i=m}^{\infty}(a_i + b_i) = \sum_{i=m}^{\infty} a_i + \sum_{i=m}^{\infty} b_i.$$

Exercise 2.2.3. Given an infinite series $\sum_{i=m}^{\infty} a_i$ and an integer $k \geq m$, show that $\sum_{i=m}^{\infty} a_i$ converges if and only if $\sum_{i=k}^{\infty} a_i$ converges.

Proposition 2.2.1. Suppose $\sum_{i=m}^{\infty} a_i$ converges. Then $\lim_{n\to\infty} a_n = 0$.

Proof. Let $s_n = \sum_{i=m}^{n} a_i$ and $s = \lim_{n\to\infty} s_n$. Since $a_n = s_n - s_{n-1}$, we have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}(s_n - s_{n-1}) = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = s - s = 0. \tag{2.2.4}$$

Q.E.D.

Exercise 2.2.4. Let $s = \sum_{i=0}^{\infty}(-1)^n$. Note that

$$s = \sum_{n=0}^{\infty}(-1)^n = 1 - \sum_{n=0}^{\infty}(-1)^n = 1 - s,$$

from which it follows that $s = \frac{1}{2}$. Is this correct?

Exercise 2.2.5. Show that for any real number $x \neq 1$,

$$s_n = \sum_{i=0}^{n} x^i = \frac{1 - x^{n+1}}{1 - x}.$$

(Hint: Note that $x^{n+1} = s_{n+1} - s_n = 1 + xs_n - s_n$.)

Proposition 2.2.2. For any real number x with $|x| < 1$,

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}. \tag{2.2.5}$$

Proof. If $s_n = \sum_{i=0}^{n} x^i$, then, by the previous exercise,

$$s_n = \frac{1 - x^{n+1}}{1 - x}. \tag{2.2.6}$$

Hence

$$\sum_{n=0}^{\infty} x^n = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{1 - x^{n+1}}{1 - x} = \frac{1}{1-x}. \tag{2.2.7}$$

Q.E.D.

2.2.1 Comparison tests

The following two propositions are together referred to as the *comparison test*.

Proposition 2.2.3. Suppose $\sum_{i=m}^{\infty} a_i$ and $\sum_{i=k}^{\infty} b_i$ are infinite series for which there exists an integer N such that $0 \leq a_i \leq b_i$ whenever $i \geq N$. If $\sum_{i=k}^{\infty} b_i$ converges, then $\sum_{i=m}^{\infty} a_i$ converges.

Proof. By Exercise 2.2.3 We need only show that $\sum_{i=N}^{\infty} a_i$ converges. Let s_n be the nth partial sum of $\sum_{i=N}^{\infty} a_i$ and let t_n be the nth partial sum of $\sum_{i=N}^{\infty} b_i$. Now

$$s_{n+1} - s_n = a_{n+1} \geq 0 \tag{2.2.8}$$

for every $n \geq N$, so $\{s_n\}_{n=N}^{\infty}$ is a nondecreasing sequence. Moreover,

$$s_n \leq t_n \leq \sum_{i=N}^{\infty} b_i < +\infty \tag{2.2.9}$$

for every $n \geq N$. Thus $\{s_n\}_{n=N}^{\infty}$ is a nondecreasing, bounded sequence, and so converges. **Q.E.D.**

Proposition 2.2.4. Suppose $\sum_{i=m}^{\infty} a_i$ and $\sum_{i=k}^{\infty} b_i$ are infinite series for which there exists an integer N such that $0 \leq a_i \leq b_i$ whenever $i \geq N$. If $\sum_{i=k}^{\infty} a_i$ diverges, then $\sum_{i=m}^{\infty} b_i$ diverges.

Proof. By Exercise 2.2.3 we need only show that $\sum_{i=N}^{\infty} b_i$ diverges. Let s_n be the nth partial sum of $\sum_{i=N}^{\infty} a_i$ and let t_n be the nth partial sum of $\sum_{i=N}^{\infty} b_i$. Now $\{s_n\}_{n=N}^{\infty}$ is a nondecreasing sequence which diverges, and so we must have $\lim_{n \to \infty} s_n = +\infty$. Thus given any real number M there exists an integer L such that

$$M < s_n \leq t_n \tag{2.2.10}$$

whenever $n > L$. Hence $\lim_{n \to \infty} t_n = +\infty$ and $\sum_{i=m}^{\infty} b_i$ diverges. **Q.E.D.**

Example 2.2.1. Consider the infinite series

$$\sum_{n=0}^{\infty} \frac{1}{n!} = 1 + 1 + \frac{1}{2} + \frac{1}{3!} + \frac{1}{4!} + \cdots .$$

Now for $n = 1, 2, 3, \ldots$, we have

$$0 < \frac{1}{n!} \leq \frac{1}{2^{n-1}}.$$

Since

$$\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$$

converges, it follows that

$$\sum_{n=0}^{\infty} \frac{1}{n!}$$

converges. Moreover,

$$2 < \sum_{n=0}^{\infty} \frac{1}{n!} < 1 + \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = 1 + \frac{1}{1 - \frac{1}{2}} = 3.$$

We let

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}. \tag{2.2.11}$$

Proposition 2.2.5. $e \notin \mathbb{Q}$.

Proof. Suppose $e = \frac{p}{q}$ where $p, q \in \mathbb{Z}^+$. Let

$$a = q!\left(e - \sum_{i=0}^{q} \frac{1}{n!}\right). \tag{2.2.12}$$

Then $a \in \mathbb{Z}^+$ since $q!e = (q-1)!p$ and $n!$ divides $q!$ when $n \leq q$. At the same time

$$\begin{aligned}
a &= q!\left(\sum_{n=0}^{\infty}\frac{1}{n!}-\sum_{i=0}^{q}\frac{1}{n!}\right)\\
&= q!\sum_{n=q+1}^{\infty}\frac{1}{n!}\\
&= \left(\frac{1}{q+1}+\frac{1}{(q+1)(q+2)}+\frac{1}{(q+1)(q+2)(q+3)}+\cdots\right)\\
&= \frac{1}{q+1}\left(1+\frac{1}{q+2}+\frac{1}{(q+2)(q+3)}+\cdots\right)\\
&< \frac{1}{q+1}\left(1+\frac{1}{q+1}+\frac{1}{(q+1)^2}+\cdots\right)\\
&= \frac{1}{q+1}\sum_{n=0}^{\infty}\frac{1}{(q+1)^n}\\
&= \frac{1}{q+1}\left(\frac{1}{1-\frac{1}{q+1}}\right)\\
&= \frac{1}{q}. \qquad (2.2.13)
\end{aligned}$$

Since this is impossible, we conclude that no such integers p and q exist. **Q.E.D.**

Definition 2.2.2. We call a real number which is not a rational number an *irrational* number.

Example 2.2.2. We have seen that $\sqrt{2}$ and e are irrational numbers.

Proposition 2.2.6. Suppose $\sum_{i=m}^{\infty} a_i$ and $\sum_{i=k}^{\infty} b_i$ are infinite series for which there exists an integer N and a real number $M > 0$ such that $0 \leq a_i \leq Mb_i$ whenever $i \geq N$. If $\sum_{i=k}^{\infty} b_i$ converges, then $\sum_{i=m}^{\infty} a_i$ converges.

Proof. Since $\sum_{i=k}^{\infty} Mb_i$ converges whenever $\sum_{i=k}^{\infty} b_i$ does, the result follows from the comparison test. **Q.E.D.**

Exercise 2.2.6. Suppose $\sum_{i=m}^{\infty} a_i$ diverges. Show that $\sum_{i=m}^{\infty} \beta a_i$ diverges for any real number $\beta \neq 0$.

Proposition 2.2.7. Suppose $\sum_{i=m}^{\infty} a_i$ and $\sum_{i=k}^{\infty} b_i$ are infinite series for which there exists an integer N and a real number $M > 0$ such that $0 \leq a_i \leq M b_i$ whenever $i \geq N$. If $\sum_{i=m}^{\infty} a_i$ diverges, then $\sum_{i=k}^{\infty} b_i$ diverges.

Proof. By the comparison test, $\sum_{i=m}^{\infty} M b_i$ diverges. Hence, by the previous exercise, $\sum_{i=m}^{\infty} b_i$ also diverges. **Q.E.D.**

We call the results of the next two exercises, which are direct consequences of the last two propositions, the *limit comparison test*.

Exercise 2.2.7. Suppose $\sum_{i=m}^{\infty} a_i$ and $\sum_{i=m}^{\infty} b_i$ are infinite series for which $a_i \geq 0$ and $b_i > 0$ for all $i \geq m$. Show that if $\sum_{i=m}^{\infty} b_i$ converges and

$$\lim_{i \to \infty} \frac{a_i}{b_i} < +\infty, \tag{2.2.14}$$

then $\sum_{i=m}^{\infty} a_i$ converges.

Exercise 2.2.8. Suppose $\sum_{i=m}^{\infty} a_i$ and $\sum_{i=m}^{\infty} b_i$ are infinite series for which $a_i \geq 0$ and $b_i > 0$ for all $i \geq m$. Show that if $\sum_{i=m}^{\infty} a_i$ diverges and

$$\lim_{i \to \infty} \frac{a_i}{b_i} < +\infty, \tag{2.2.15}$$

then $\sum_{i=m}^{\infty} b_i$ diverges.

Exercise 2.2.9. Show that

$$\sum_{n=1}^{\infty} \frac{1}{n 2^n}$$

converges.

Exercise 2.2.10. Show that

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

converges for any real number $x \geq 0$.

Chapter 3

Cardinality

3.1 Binary representations

Suppose $\{a_n\}_{n=1}^{\infty}$ is a sequence such that, for each $n = 1, 2, 3, \ldots$, either $a_n = 0$ or $a_n = 1$ and, for any integer N, there exists an integer $n > N$ such that $a_n = 0$. Then

$$0 \leq \frac{a_n}{2^n} \leq \frac{1}{2^n} \tag{3.1.1}$$

for $n = 1, 2, 3, \ldots$, so the infinite series

$$\sum_{n=1}^{\infty} \frac{a_n}{2^n} \tag{3.1.2}$$

converges to some real number x by the comparison test. Moreover,

$$0 \leq x < \sum_{n=1}^{\infty} \frac{1}{2^n} = 1. \tag{3.1.3}$$

We call the sequence $\{a_n\}_{n=1}^{\infty}$ the *binary representation* for x, and write

$$x = .a_1 a_2 a_3 a_4 \ldots. \tag{3.1.4}$$

Exercise 3.1.1. Suppose $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are both binary representations for x. Show that $a_n = b_n$ for $n = 1, 2, 3, \ldots$.

Now suppose $x \in \mathbb{R}$ with $0 \leq x < 1$. Construct a sequence $\{a_n\}_{n=1}^{\infty}$ as follows: If $0 \leq x < \frac{1}{2}$, let $a_1 = 0$; otherwise, let $a_1 = 1$. For $n = 1, 2, 3, \ldots$, let

$$s_n = \sum_{i=1}^{n} \frac{a_i}{2^i} \tag{3.1.5}$$

and set $a_{n+1} = 1$ if

$$s_n + \frac{1}{2^{n+1}} \leq x \tag{3.1.6}$$

and $a_{n+1} = 0$ otherwise.

Lemma 3.1.1. With the notation as above,

$$s_n \leq x < s_n + \tfrac{1}{2^n} \tag{3.1.7}$$

for $n = 1, 2, 3, \ldots$.

Proof. Since

$$s_1 = \begin{cases} 0, & \text{if } 0 \leq x < \dfrac{1}{2}, \\ \dfrac{1}{2}, & \text{if } \dfrac{1}{2} \leq x < 1, \end{cases} \tag{3.1.8}$$

it is clear that $s_1 \leq x < s_1 + \frac{1}{2}$. So suppose $n > 1$ and $s_{n-1} \leq x < s_{n-1} + \frac{1}{2^{n-1}}$. If $s_{n-1} + \frac{1}{2^n} \leq x$, then $a_n = 1$ and

$$s_n = s_{n-1} + \frac{1}{2^n} \leq x < s_{n-1} + \frac{1}{2^{n-1}} = s_{n-1} + \frac{1}{2^n} + \frac{1}{2^n} = s_n + \frac{1}{2^n}. \tag{3.1.9}$$

If $x < s_{n-1} + \frac{1}{2^n}$, then $a_n = 0$ and

$$s_n = s_{n-1} \leq x < s_{n-1} + \frac{1}{2^n} = s_n + \frac{1}{2^n}. \tag{3.1.10}$$

Q.E.D.

Proposition 3.1.2. With the notation as above,

$$x = \sum_{n=1}^{\infty} \frac{a_n}{2^n}. \tag{3.1.11}$$

Proof. Given $\epsilon > 0$, choose an integer N such that $\frac{1}{2^N} < \epsilon$. Then, for any $n > N$, it follows from the lemma that

$$|s_n - x| < \frac{1}{2^n} < \frac{1}{2^N} < \epsilon. \tag{3.1.12}$$

Hence

$$x = \lim_{n \to \infty} s_n = \sum_{n=1}^{\infty} \frac{a_n}{2^n}. \tag{3.1.13}$$

Q.E.D.

Lemma 3.1.3. With the notation as above, given any integer N there exists an integer $n > N$ such that $a_n = 0$.

Proof. If $a_n = 1$ for $n = 1, 2, 3, \ldots$, then

$$x = \sum_{n=1}^{\infty} \frac{1}{2^n} = 1, \tag{3.1.14}$$

contradicting the assumption that $0 \leq x < 1$. Now suppose there exists an integer N such that $a_N = 0$ but $a_n = 1$ for every $n > N$. Then

$$x = s_N + \sum_{n=N+1}^{\infty} \frac{1}{2^n} = s_{N-1} + \sum_{n=N+1}^{\infty} \frac{1}{2^n} = s_{N-1} + \frac{1}{2^N}, \tag{3.1.15}$$

implying that $a_N = 1$, and thus contradicting the assumption that $a_N = 0$. **Q.E.D.**

Combining the previous lemma with the previous proposition yields the following result.

Proposition 3.1.4. With the notation as above, $x = .a_1a_2a_3a_4\ldots$.

The next theorem now follows from Exercise 3.1.1 and the previous proposition.

Theorem 3.1.5. Every real number $0 \leq x < 1$ has a unique binary representation.

3.2 Countable and uncountable sets

Definition 3.2.1. A function $\varphi : A \to B$ is said to be a *one-to-one correspondence* if φ is both one-to-one and onto.

Definition 3.2.2. We say sets A and B have the same *cardinality* if there exists a one-to-one correspondence $\varphi : A \to B$.

We denote the fact that A and B have the same cardinality by writing $|A| = |B|$.

Exercise 3.2.1. Define a relation on sets by setting $A \sim B$ if and only if $|A| = |B|$. Show that this relation is an equivalence relation.

Definition 3.2.3. Let A be a set. If, for $n \in \mathbb{Z}^+$, A has the cardinality of the set $\{1, 2, 3, \ldots, n\}$, we say A is *finite* and write $|A| = n$. If A has the cardinality of $\mathbb{Z}^+$, we say A is *countable* and write $|A| = \aleph_0$.

Example 3.2.1. If we define $\varphi : \mathbb{Z}^+ \to \mathbb{Z}$ by

$$\varphi(n) = \begin{cases} \dfrac{n-1}{2}, & \text{if } n \text{ is odd,} \\ -\dfrac{n}{2}, & \text{if } n \text{ is even,} \end{cases} \tag{3.2.1}$$

then φ is a one-to-one correspondence. Thus $|\mathbb{Z}| = \aleph_0$.

Exercise 3.2.2. Let A be the set of even integers. Show that $|A| = \aleph_0$.

Exercise 3.2.3. Verify each of the following:

a. If A is a nonempty subset of $\mathbb{Z}^+$, then A is either finite or countable.

b. If A is a nonempty subset of a countable set B, then A is either finite or countable.

Proposition 3.2.1. Suppose A and B are countable sets. Then the set $C = A \cup B$ is countable.

Proof. Suppose A and B are disjoint, that is, $A \cap B = \emptyset$. Let $\varphi : \mathbb{Z}^+ \to A$ and $\psi : \mathbb{Z}^+ \to B$ be one-to-one correspondences. Define $\tau : \mathbb{Z}^+ \to C$ by

$$\tau(n) = \begin{cases} \varphi\left(\frac{n+1}{2}\right), & \text{if } n \text{ is odd,} \\ \psi\left(\frac{n}{2}\right), & \text{if } n \text{ is even.} \end{cases} \tag{3.2.2}$$

Then τ is a one-to-one correspondence, showing that C is countable.

If A and B are not disjoint, then τ is onto but not one-to-one. However, in that case C has the cardinality of an infinite subset of $\mathbb{Z}^+$, and so is countable. **Q.E.D.**

Definition 3.2.4. A nonempty set which is not finite is said to be *infinite*. An infinite set which is not countable is said to be *uncountable*.

Exercise 3.2.4. Suppose A is uncountable and $B \subset A$ is countable. Show that $A \setminus B$ is uncountable.

Proposition 3.2.2. Suppose A and B are countable. Then $C = A \times B$ is countable.

Proof. Let $\varphi : \mathbb{Z}^+ \to A$ and $\psi : \mathbb{Z}^+ \to B$ be one-to-one correspondences. Let $a_i = \varphi(i)$ and $b_i = \psi(i)$. Define $\tau : \mathbb{Z}^+ \to C$ by letting

$$\tau(1) = (a_1, b_1), \tag{3.2.3}$$
$$\tau(2) = (a_1, b_2), \tag{3.2.4}$$
$$\tau(3) = (a_2, b_1), \tag{3.2.5}$$
$$\tau(4) = (a_1, b_3), \tag{3.2.6}$$
$$\tau(5) = (a_2, b_2), \tag{3.2.7}$$
$$\tau(6) = (a_3, b_1), \tag{3.2.8}$$
$$\tau(7) = (a_1, b_4), \tag{3.2.9}$$
$$\vdots \;=\; \vdots \tag{3.2.10}$$

That is, form the infinite matrix with (a_i, b_j) in the ith row and jth column, and then count the entries by reading down the diagonals from right to left. Then τ is a one-to-one correspondence and C is countable. **Q.E.D.**

Proposition 3.2.3. $\mathbb{Q}$ is countable.

Proof. By the previous proposition, $\mathbb{Z} \times \mathbb{Z}$ is countable. Let

$$A = \{(p, q) : p, q \in \mathbb{Z}, q > 0, p \text{ and } q \text{ relatively prime}\}. \tag{3.2.11}$$

Then A is infinite and $A \subset \mathbb{Z} \times \mathbb{Z}$, so A is countable. But clearly $|\mathbb{Q}| = |A|$, so $\mathbb{Q}$ is countable. **Q.E.D.**

Proposition 3.2.4. Suppose for each $i \in \mathbb{Z}^+$, A_i is countable. Then

$$B = \bigcup_{i=1}^{\infty} A_i \tag{3.2.12}$$

is countable.

Proof. Suppose the sets A_i, $i \in \mathbb{Z}^+$, are pairwise disjoint, that is, $A_i \cap A_j = \emptyset$ for all $i, j \in \mathbb{Z}^+$. For each $i \in \mathbb{Z}^+$, let $\varphi_i : \mathbb{Z}^+ \to A_i$ be a one-to-one correspondence. Then $\psi : \mathbb{Z}^+ \times \mathbb{Z}^+ \to B$ defined by

$$\psi(i, j) = \varphi_i(j) \tag{3.2.13}$$

is a one-to-one correspondence, and so $|B| = |\mathbb{Z}^+ \times \mathbb{Z}^+| = \aleph_0$.

If the sets A_i, $i \in \mathbb{Z}^+$, are not disjoint, then ψ is onto but not one-to-one. But then there exists a subset P of $\mathbb{Z}^+ \times \mathbb{Z}^+$ such that $\psi : P \to B$ is a one-to-one correspondence. Since P is an infinite subset of a countable set, P is countable and so $|B| = \aleph_0$. **Q.E.D.**

If in the previous proposition we allow that, for each $i \in \mathbb{Z}^+$, A_i is either finite or countable, then $B = \bigcup_{i=1}^{\infty} A_i$ will be either finite or countable.

3.3 Power sets

Definition 3.3.1. Given a set A, we call the set of all subsets of A the *power set* of A, which we denote $\mathcal{P}(A)$.

Example 3.3.1. If $A = \{1, 2, 3\}$, then

$$\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}.$$

Proposition 3.3.1. If A is finite with $|A| = n$, then $|\mathcal{P}(A)| = 2^n$.

Proof. Let

$$B = \{(b_1, b_2, \ldots, b_n) : b_i = 0 \text{ or } b_i = 1, i = 1, 2, \ldots, n\} \tag{3.3.1}$$

and let $a_1, a_2, \ldots, a_n$ be the elements of A. Define $\varphi : B \to \mathcal{P}(A)$ by letting

$$\varphi(b_1, b_2, \ldots, b_n) = \{a_i : b_i = 1, i = 1, 2, \ldots, n\}. \tag{3.3.2}$$

Then φ is a one-to-one correspondence. The result now follows from the next exercise. **Q.E.D.**

Exercise 3.3.1. With B as in the previous proposition, show that $|B| = 2^n$.

In analogy with the case when A is finite, we let $2^{|A|} = |\mathcal{P}(A)|$ for any nonempty set A.

Definition 3.3.2. Suppose A and B are sets for which there exists a one-to-one function $\varphi : A \to B$ but there does not exist a one-to-one correspondence $\psi : A \to B$. Then we write $|A| < |B|$.

Theorem 3.3.2. If A is a nonempty set, then $|A| < |\mathcal{P}(A)|$.

Proof. Define $\varphi : A \to \mathcal{P}(A)$ by $\varphi(a) = \{a\}$. Then φ is one-to-one. Now suppose $\psi : A \to \mathcal{P}(A)$ is any one-to-one function. Let

$$C = \{a : a \in A, a \notin \psi(a)\}. \tag{3.3.3}$$

Suppose there exists $a \in A$ such that $\psi(a) = C$. Then $a \in C$ if and only if $a \notin C$, an obvious contradiction. Thus C is not in the range of ψ, and so ψ is not a one-to-one correspondence. **Q.E.D.**

Lemma 3.3.3. Let A be the set of all sequences $\{a_i\}_{i=1}^{\infty}$ with $a_i = 0$ or $a_i = 1$ for each $i = 1, 2, 3, \ldots$. Then $|A| = |\mathcal{P}(\mathbb{Z}^+)|$.

Proof. Define $\varphi : A \to \mathcal{P}(\mathbb{Z}^+)$ by

$$\varphi\left(\{a_i\}_{i=1}^{\infty}\right) = \{i : i \in \mathbb{Z}^+, a_i = 1\}. \tag{3.3.4}$$

Then φ is a one-to-one correspondence. **Q.E.D.**

Now let B be the set of all sequences $\{a_i\}_{i=1}^{\infty}$ in A such that for every integer N there exists an integer $n > N$ such that $a_n = 0$. Let $C = A \setminus B$,

$$D_0 = \{\{a_i\}_{i=1}^{\infty} : a_i = 1, i = 1, 2, 3, \ldots\}, \tag{3.3.5}$$

and

$$D_j = \{\{a_i\}_{i=1}^{\infty} : a_j = 0, a_k = 1 \text{ for } k > j\} \tag{3.3.6}$$

for $j = 1, 2, 3, \ldots$. Then $|D_0| = 1$ and $|D_j| = 2^{j-1}$ for $j = 1, 2, 3, \ldots$. Moreover,

$$C = \bigcup_{j=0}^{\infty} D_j, \tag{3.3.7}$$

so C is countable. Since $A = B \cup C$, and A is uncountable, it follows that B is uncountable. Now if we let

$$I = \{x : x \in \mathbb{R}, 0 \leq x < 1\}, \tag{3.3.8}$$

we have seen that the function $\varphi : B \to I$ defined by

$$\varphi\left(\{a_i\}_{i=1}^{\infty}\right) = .a_1 a_2 a_3 a_4 \dots \tag{3.3.9}$$

is a one-to-one correspondence. It follows that I is uncountable. As a consequence, we have the following result.

Theorem 3.3.4. $\mathbb{R}$ is uncountable.

Exercise 3.3.2. Let $I = \{x : x \in \mathbb{R}, 0 \leq x < 1\}$. Show that

a. $|I| = |\{x : x \in \mathbb{R}, 0 \leq x \leq 1\}|$

b. $|I| = |\{x : x \in \mathbb{R}, 0 < x < 1\}|$

c. $|I| = |\{x : x \in \mathbb{R}, 0 < x < 2\}|$

d. $|I| = |\{x : x \in \mathbb{R}, -1 < x < 1\}|$

Exercise 3.3.3. Let $I = \{x : x \in \mathbb{R}, 0 \leq x < 1\}$ and suppose a and b are real numbers with $a < b$. Show that

$$|I| = |\{x : x \in \mathbb{R}, a \leq x < b\}|.$$

Exercise 3.3.4. Does there exist a set $A \subset \mathbb{R}$ for which $\aleph_0 < |A| < 2^{\aleph_0}$? (Before working too long on this problem, you may wish to read about Cantor's continuum hypothesis.)

Chapter 4

Topology of the Real Line

4.1 Intervals

Definition 4.1.1. Given any two extended real numbers $a < b$, we call the set

$$(a, b) = \{x : x \in \mathbb{R}, a < x < b\} \tag{4.1.1}$$

an *open interval*. Given any two finite real numbers $a \leq b$, we call the sets

$$[a, b] = \{x : x \in \mathbb{R}, a \leq x \leq b\}, \tag{4.1.2}$$

$$(-\infty, b] = \{x : x \in \mathbb{R}, x \leq b\}, \tag{4.1.3}$$

and

$$[a, +\infty) = \{x : x \in \mathbb{R}, x \geq a\} \tag{4.1.4}$$

closed intervals. We call any set which is an open interval, a closed interval, or is given by, for some finite real numbers $a < b$,

$$(a, b] = \{x : x \in \mathbb{R}, a < x \leq b\} \tag{4.1.5}$$

or

$$[a, b) = \{x : x \in \mathbb{R}, a \leq x < b\}, \tag{4.1.6}$$

an *interval*.

Proposition 4.1.1. If $a, b \in \mathbb{R}$ with $a < b$, then

$$(a, b) = \{x : x = \lambda a + (1 - \lambda)b, 0 < \lambda < 1\}. \tag{4.1.7}$$

Proof. Suppose $x = \lambda a + (1 - \lambda)b$ for some $0 < \lambda < 1$. Then

$$b - x = \lambda b - \lambda a = \lambda(b - a) > 0, \tag{4.1.8}$$

so $x < b$. Similarly,

$$x - a = (\lambda - 1)a + (1 - \lambda)b = (1 - \lambda)(b - a) > 0, \tag{4.1.9}$$

so $a < x$. Hence $x \in (a, b)$.

Now suppose $x \in (a, b)$. Then

$$x = \left(\frac{b-x}{b-a}\right) a + \left(\frac{x-a}{b-a}\right) b = \left(\frac{b-x}{b-a}\right) a + \left(1 - \frac{b-x}{b-a}\right) b \tag{4.1.10}$$

and

$$0 < \frac{b-x}{b-a} < 1. \tag{4.1.11}$$

Q.E.D.

4.2 Open sets

Definition 4.2.1. We say a set $U \subset \mathbb{R}$ is *open* if for every $x \in U$ there exists $\epsilon > 0$ such that

$$(x - \epsilon, x + \epsilon) \subset U. \tag{4.2.1}$$

Proposition 4.2.1. Every open interval I is an open set.

Proof. Suppose $I = (a, b)$, where $a < b$ are extended real numbers. Given $x \in I$, let ϵ be the smaller of $x - a$ and $b - x$. Suppose $y \in (x - \epsilon, x + \epsilon)$. If $b = +\infty$, then $b > y$; otherwise, we have

$$b - y > b - (x + \epsilon) = (b - x) - \epsilon \geq (b - x) - (b - x) = 0, \tag{4.2.2}$$

so $b > y$. If $a = -\infty$, then $a < y$; otherwise,

$$y - a > (x - \epsilon) - a = (x - a) - \epsilon \geq (x - a) - (x - a) = 0, \tag{4.2.3}$$

so $a < y$. Thus $y \in I$ and I is an open set. **Q.E.D.**

Note that $\mathbb{R}$ is an open set (it is, in fact, the open interval $(-\infty, +\infty)$), as is $\emptyset$ (it satisfies the definition trivially).

Proposition 4.2.2. Suppose A is a set and, for each $\alpha \in A$, U_α is an open set. Then

$$\bigcup_{\alpha \in A} U_\alpha \tag{4.2.4}$$

is an open set.

Proof. Let $x \in \bigcup_{\alpha \in A} U_\alpha$. Then $x \in U_\alpha$ for some $\alpha \in A$. Since U_α is open, there exists an $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subset U_\alpha$. Thus

$$(x - \epsilon, x + \epsilon) \subset U_\alpha \subset \bigcup_{\alpha \in A} U_\alpha. \tag{4.2.5}$$

Hence $\bigcup_{\alpha \in A} U_\alpha$ is open. **Q.E.D.**

Proposition 4.2.3. Suppose $U_1, U_2, \ldots, U_n$ is a finite collection of open sets. Then

$$\bigcap_{i=1}^{n} U_i \tag{4.2.6}$$

is open.

Proof. Let $x \in \bigcap_{i=1}^{n} U_i$. Then $x \in U_i$ for every $i = 1, 2, \ldots, n$. For each i, choose $\epsilon_i > 0$ such that $(x - \epsilon_i, x + \epsilon_i) \subset U_i$. Let ϵ be the smallest of $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$. Then $\epsilon > 0$ and

$$(x - \epsilon, x + \epsilon) \subset (x - \epsilon_i, x + \epsilon_i) \subset U_i \tag{4.2.7}$$

for every $i = 1, 2, \ldots, n$. Thus

$$(x - \epsilon, x + \epsilon) \subset \bigcap_{i=1}^{n} U_i. \tag{4.2.8}$$

Hence $\bigcap_{i=1}^{n} U_i$ is an open set. **Q.E.D.**

Along with the facts that $\mathbb{R}$ and $\emptyset$ are both open sets, the last two propositions show that the collection of open subsets of $\mathbb{R}$ form a *topology*.

Definition 4.2.2. Let $A \subset \mathbb{R}$. We say $x \in A$ is an *interior point* of A if there exists an $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subset A$. We call the set of all interior points of A the *interior* of A, denoted A°.

Exercise 4.2.1. Show that if $A \subset \mathbb{R}$, then A° is open.

Exercise 4.2.2. Show that A is open if and only if $A = A^\circ$.

Exercise 4.2.3. Let $U \subset \mathbb{R}$ be a nonempty open set. Show that $\sup U \notin U$ and $\inf U \notin U$.

4.3 Closed sets

Definition 4.3.1. We call a point $x \in \mathbb{R}$ a *limit point* of a set $A \subset \mathbb{R}$ if for every $\epsilon > 0$ there exists $a \in A$, $a \neq x$, such that $a \in (x - \epsilon, x + \epsilon)$.

Definition 4.3.2. Suppose $A \subset \mathbb{R}$. We call a point $a \in A$ an *isolated point* of A if there exists an $\epsilon > 0$ such that

$$A \cap (a - \epsilon, a + \epsilon) = \{a\}. \tag{4.3.1}$$

Exercise 4.3.1. Identify the limit points and isolated points of the following sets:

a. $[-1, 1]$,

b. $(-1, 1)$,

c. $\left\{\frac{1}{n} : n \in \mathbb{Z}^+\right\}$,

d. $\mathbb{Z}$,

e. $\mathbb{Q}$.

Exercise 4.3.2. Suppose x is a limit point of the set A. Show that for every $\epsilon > 0$, the set $(x - \epsilon, x + \epsilon) \cap A$ is infinite.

We let A' denote the set of limit points of a set A.

Definition 4.3.3. Given a set $A \subset \mathbb{R}$, we call the set $\overline{A} = A \cup A'$ the *closure* of A.

Definition 4.3.4. We call a set $C \subset \mathbb{R}$ *closed* if $C = \overline{C}$.

Proposition 4.3.1. If $A \subset \mathbb{R}$, then $\overline{A}$ is closed.

Proof. Suppose x is a limit point of $\overline{A}$. We will show that x is a limit point of A, and hence $x \in \overline{A}$. Now for any $\epsilon > 0$, there exists $a \in \overline{A}$, $a \neq x$, such that

$$a \in \left(x - \frac{\epsilon}{2}, x + \frac{\epsilon}{2}\right). \tag{4.3.2}$$

If $a \in A$, let $b = a$. If $a \notin A$, then a is a limit point of A, so there exists $b \in A$, $b \neq a$ and $b \neq x$, such that

$$b \in \left(a - \frac{\epsilon}{2}, a + \frac{\epsilon}{2}\right). \tag{4.3.3}$$

In either case

$$|x - b| \leq |x - a| + |a - b| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \tag{4.3.4}$$

Hence $x \in A'$, and so $\overline{A}$ is closed. **Q.E.D.**

Proposition 4.3.2. A set $C \subset \mathbb{R}$ is closed if and only if for every convergent sequence $\{a_k\}_{k \in K}$ with $a_k \in C$ for all $k \in K$,

$$\lim_{k \to \infty} a_k \in C. \tag{4.3.5}$$

Proof. Suppose C is closed and $\{a_k\}_{k\in K}$ is a convergent sequence with $a_k \in C$ for all $k \in K$. Let $x = \lim_{k\to\infty} a_k$. If $x = a_k$ for some integer k, then $x \in C$. Otherwise, for every $\epsilon > 0$, there exists an integer N such that $|a_N - x| < \epsilon$. Hence $a_N \neq x$ and

$$a_N \in (x - \epsilon, x + \epsilon). \tag{4.3.6}$$

Thus x is a limit point of C, and so $x \in C$ since C is closed.

Now suppose that for every convergent sequence $\{a_k\}_{k\in K}$ with $a_k \in C$ for all $k \in K$, $\lim_{k\to\infty} a_k \in C$. Let x be a limit point of C. For $k = 1, 2, 3, \ldots$, choose $a_k \in C$ such that $a_k \in \left(x - \frac{1}{k}, x + \frac{1}{k}\right)$. Then clearly

$$x = \lim_{k\to\infty} a_k, \tag{4.3.7}$$

so $x \in C$. Thus C is closed. **Q.E.D.**

Exercise 4.3.3. Show that every closed interval I is a closed set.

Proposition 4.3.3. Suppose A is a set and, for each $\alpha \in A$, C_α is a closed set. Then

$$\bigcap_{\alpha\in A} C_\alpha \tag{4.3.8}$$

is a closed set.

Proof. Suppose x is a limit point of $\bigcap_{\alpha\in A} C_\alpha$. Then for any $\epsilon > 0$, there exists $y \in \bigcap_{\alpha\in A} C_\alpha$ such that $y \neq x$ and $y \in (x - \epsilon, x + \epsilon)$. But then for any $\alpha \in A$, $y \in C_\alpha$, so x is a limit point of C_α. Since C_α is closed, it follows that $x \in C_\alpha$ for every $\alpha \in A$. Thus $x \in \bigcap_{\alpha\in A} C_\alpha$ and $\bigcap_{\alpha\in A} C_\alpha$ is closed. **Q.E.D.**

Proposition 4.3.4. Suppose $C_1, C_2, \ldots, C_n$ is a finite collection of closed sets. Then

$$\bigcup_{i=1}^{n} C_i \tag{4.3.9}$$

is closed.

Proof. Suppose $\{a_k\}_{k\in K}$ is a convergent sequence with $a_k \in \bigcup_{i=1}^n C_i$ for every $k \in K$. Let $L = \lim_{k\to\infty} a_k$. Since K is an infinite set, there must exist an integer m and a subsequence $\{a_{n_j}\}_{j=1}^\infty$ such that $a_{n_j} \in C_m$ for $j = 1, 2, \ldots$. Since every subsequence of $\{a_k\}_{k\in K}$ converges to L, $\{a_{n_j}\}_{j=1}^\infty$ must converge to L. Since C_m is closed,

$$L = \lim_{j\to\infty} a_{n_j} \in C_m \subset \bigcup_{i=1}^{n} C_i. \tag{4.3.10}$$

Thus $\bigcup_{i=1}^n C_i$ is closed. **Q.E.D.**

Note that both $\mathbb{R}$ and $\emptyset$ satisfy the definition of a closed set.

Proposition 4.3.5. A set $C \subset \mathbb{R}$ is closed if and only if $\mathbb{R} \setminus C$ is open.

Proof. Assume C is closed and let $U = \mathbb{R} \setminus C$. If $C = \mathbb{R}$, then $U = \emptyset$, which is open; if $C = \emptyset$, then $U = \mathbb{R}$, which is open. So we may assume both C and U are nonempty. Let $x \in U$. Then x is not a limit point of C, so there exists an $\epsilon > 0$ such that

$$(x - \epsilon, x + \epsilon) \cap C = \emptyset. \tag{4.3.11}$$

Thus

$$(x - \epsilon, x + \epsilon) \subset U, \tag{4.3.12}$$

so U is open.

Now suppose $U = \mathbb{R} \setminus C$ is open. If $U = \mathbb{R}$, then $C = \emptyset$, which is closed; if $U = \emptyset$, then $C = \mathbb{R}$, which is closed. So we may assume both U and C are nonempty. Let x be a limit point of C. Then, for every $\epsilon > 0$,

$$(x - \epsilon, x + \epsilon) \cap C \neq \emptyset. \tag{4.3.13}$$

Hence there does not exist $\epsilon > 0$ such that

$$(x - \epsilon, x + \epsilon) \subset U. \tag{4.3.14}$$

Thus $x \notin U$, so $x \in C$ and C is closed. **Q.E.D.**

Exercise 4.3.4. For $n = 1, 2, 3, \ldots$, let $I_n = \left(-\frac{1}{n}, \frac{n+1}{n}\right)$. Is

$$\bigcap_{n=1}^{\infty} I_n$$

open or closed?

Exercise 4.3.5. For $n = 3, 4, 5, \ldots$, let $I_n = \left[\frac{1}{n}, \frac{n-1}{n}\right]$. Is

$$\bigcup_{n=3}^{\infty} I_n$$

open or closed?

Exercise 4.3.6. Suppose, for $n = 1, 2, 3, \ldots$, the intervals $I_n = [a_n, b_n]$ are such that $I_{n+1} \subset I_n$. If $a = \sup\{a_n : n \in \mathbb{Z}^+\}$ and $b = \inf\{b_n : n \in \mathbb{Z}^+\}$, show that

$$\bigcap_{n=1}^{\infty} I_n = [a, b].$$

Exercise 4.3.7. Find a sequence I_n, $n = 1, 2, 3, \ldots$, of closed intervals such that $I_{n+1} \subset I_n$ for $n = 1, 2, 3, \ldots$ and

$$\bigcap_{n=1}^{\infty} I_n = \emptyset.$$

Exercise 4.3.8. Find a sequence I_n, $n = 1, 2, 3, \ldots$, of bounded, open intervals such that $I_{n+1} \subset I_n$ for $n = 1, 2, 3, \ldots$ and

$$\bigcap_{n=1}^{\infty} I_n = \emptyset.$$

Exercise 4.3.9. Suppose $A_i \subset \mathbb{R}$, $i = 1, 2, \ldots, n$, and let $B = \bigcup_{i=1}^{n} A_i$. Show that

$$\overline{B} = \bigcup_{i=1}^{n} \overline{A_i}.$$

Exercise 4.3.10. Suppose $A_i \subset \mathbb{R}$, $i \in \mathbb{Z}^+$, and let

$$B = \bigcup_{i=1}^{\infty} A_i.$$

Show that

$$\bigcup_{i=1}^{\infty} \overline{A_i} \subset \overline{B}.$$

Find an example for which

$$\overline{B} \neq \bigcup_{i=1}^{\infty} \overline{A_i}.$$

Exercise 4.3.11. Suppose $U \subset \mathbb{R}$ is a nonempty open set. For each $x \in U$, let

$$J_x = \bigcup (x - \epsilon, x + \delta),$$

where the union is taken over all $\epsilon > 0$ and $\delta > 0$ such that $(x - \epsilon, x + \delta) \subset U$.

a. Show that for every $x, y \in U$, either $J_x \cap J_y = \emptyset$ or $J_x = J_y$.

b. Show that

$$U = \bigcup_{x \in B} J_x,$$

where $B \subset U$ is either finite or countable.

4.4 Compact Sets

Definition 4.4.1. Suppose $T \subset \mathbb{R}$. If A is a set, U_α is an open set for every $\alpha \in A$, and

$$T \subset \bigcup_{\alpha \in A} U_\alpha, \tag{4.4.1}$$

then we call $\{U_\alpha : \alpha \in A\}$ an *open cover* of T.

Example 4.4.1. For $n = 3, 4, 5, \ldots$, let

$$U_n = \left(\frac{1}{n}, \frac{n-1}{n} \right).$$

Then $\{U_n : n = 3, 4, 5, \ldots\}$ is an open cover of the open interval $(0, 1)$.

Definition 4.4.2. Suppose $\{U_\alpha : \alpha \in A\}$ is an open cover of $T \subset \mathbb{R}$. If $B \subset A$ and

$$T \subset \bigcup_{\beta \in B} U_\beta, \tag{4.4.2}$$

then we call $\{U_\beta : \beta \in B\}$ a *subcover* of $\{U_\alpha : \alpha \in A\}$. If B is finite, we call $\{U_\beta : \beta \in B\}$ a *finite subcover* of $\{U_\alpha : \alpha \in A\}$.

Exercise 4.4.1. Show that the open cover of $(0, 1)$ given in the previous example does not have a finite subcover.

Definition 4.4.3. We say a set $K \subset \mathbb{R}$ is *compact* if every open cover of K has a finite subcover.

Example 4.4.2. As a consequence of the previous exercise, the open interval $(0, 1)$ is not compact.

Exercise 4.4.2. Show that every finite subset of $\mathbb{R}$ is compact.

Exercise 4.4.3. Suppose $n \in \mathbb{Z}^+$ and $K_1, K_2, \ldots, K_n$ are compact sets. Show that $\bigcup_{i=1}^n K_i$ and $\bigcap_{i=1}^n K_i$ are compact.

Proposition 4.4.1. If I is a closed, bounded interval, then I is compact.

Proof. Let $a \leq b$ be finite real numbers and $I = [a, b]$. Suppose $\{U_\alpha : \alpha \in A\}$ is an open cover of I. Let $\mathcal{O}$ be the set of sets $\{U_\beta : \beta \in B\}$ with the properties that B is a finite subset of A and $a \in \bigcup_{\beta \in B} U_\beta$. Let

$$T = \left\{ x : x \in I, [a, x] \subset \bigcup_{\beta \in B} U_\beta \text{ for some } \{U_\beta : \beta \in B\} \in \mathcal{O} \right\}. \tag{4.4.3}$$

Clearly, $a \in T$ so $T \neq \emptyset$. Let $s = \sup T$. Suppose $s < b$. Since $\{U_\alpha : \alpha \in A\}$ is an open cover of I, there exists an $\alpha \in A$ for which $s \in U_\alpha$. Hence there exists an $\epsilon > 0$ such that

$$(s - \epsilon, s + \epsilon) \subset U_\alpha. \tag{4.4.4}$$

Moreover, there exists a $\{U_\beta : \beta \in B\} \in \mathcal{O}$ for which

$$[a, s - \frac{\epsilon}{2}] \subset \bigcup_{\beta \in B} U_\beta. \tag{4.4.5}$$

But then

$$\{U_\beta : \beta \in B\} \cup \{U_\alpha\} \in \mathcal{O} \tag{4.4.6}$$

and

$$\left[a, s + \frac{\epsilon}{2}\right] \subset \left(\bigcup_{\beta \in B} U_\beta\right) \cup U_\alpha, \tag{4.4.7}$$

contradicting the definition of s. Hence we must have $s = b$. Now choose U_α such that $b \in U_\alpha$. Then, for some $\epsilon > 0$,

$$(b - \epsilon, b + \epsilon) \subset U_\alpha. \tag{4.4.8}$$

Moreover, there exists $\{U_\beta : \beta \in B\} \in \mathcal{O}$ such that

$$\left[a, b - \frac{\epsilon}{2}\right] \subset \bigcup_{\beta \in B} U_\beta. \tag{4.4.9}$$

Then

$$\{U_\beta : \beta \in B\} \cup \{U_\alpha\} \in \mathcal{O} \tag{4.4.10}$$

is a finite subcover of I. Thus I is compact. **Q.E.D.**

Proposition 4.4.2. If K is a closed, bounded subset of $\mathbb{R}$, then K is compact.

Proof. Since K is bounded, there exist finite real numbers a and b such that $K \subset [a, b]$. Let $\{U_\alpha : \alpha \in A\}$ be an open cover of K. Let $V = \mathbb{R} \setminus K$. Then

$$\{U_\alpha : \alpha \in A\} \cup \{V\} \tag{4.4.11}$$

is an open cover of $[a, b]$. Since $[a, b]$ is compact, there exists a finite subcover of this cover. This subcover is either of the form $\{U_\beta : \beta \in B\}$ or $\{U_\beta : \beta \in B\} \cup \{V\}$ for some $B \subset A$. In the former case, we have

$$K \subset [a, b] \subset \bigcup_{\beta \in B} U_\beta; \tag{4.4.12}$$

in the latter case, we have

$$K \subset [a, b] \setminus V \subset \bigcup_{\beta \in B} U_\beta. \tag{4.4.13}$$

In either case, we have found a finite subcover of $\{U_\alpha : \alpha \in A\}$. **Q.E.D.**

Exercise 4.4.4. Show that if K is compact and $C \subset K$ is closed, then C is compact.

Proposition 4.4.3. If $K \subset \mathbb{R}$ is compact, then K is closed.

Proof. Suppose x is a limit point of K and $x \notin K$. For $n = 1, 2, 3, \ldots$, let

$$U_n = \left(-\infty, x - \frac{1}{n}\right) \cup \left(x + \frac{1}{n}, +\infty\right). \tag{4.4.14}$$

Then

$$\bigcup_{n=1}^{\infty} U_n = (-\infty, x) \cup (x, +\infty) \supset K. \tag{4.4.15}$$

However, for any $N \in \mathbb{Z}^+$, there exists $a \in K$ with

$$a \in \left(x - \frac{1}{N}, x + \frac{1}{N}\right), \tag{4.4.16}$$

and hence

$$a \notin \bigcup_{n=1}^{N} U_n = \left(-\infty, x - \frac{1}{N}\right) \cup \left(x + \frac{1}{N}, +\infty\right). \tag{4.4.17}$$

Thus the open cover $\{U_n : n \in \mathbb{Z}^+\}$ does not have a finite subcover, contradicting the assumption that K is compact. **Q.E.D.**

Proposition 4.4.4. If $K \subset \mathbb{R}$ is compact, then K is bounded.

Proof. Suppose K is not bounded. For $n = 1, 2, 3, \ldots$, let $U_n = (-n, n)$. Then

$$\bigcup_{n=1}^{\infty} U_n = (-\infty, \infty) \supset K. \tag{4.4.18}$$

But, for any integer N, there exists $a \in K$ such that $|a| > N$, from which it follows that

$$a \notin \bigcup_{n=1}^{N} U_n = (-N, N). \tag{4.4.19}$$

Thus the open cover $\{U_n : n \in \mathbb{Z}^+\}$ does not have a finite subcover, contradicting the assumption that K is compact. **Q.E.D.**

Taken together, the previous three propositions yield the following fundamental result:

Theorem 4.4.5 (Heine-Borel Theorem)**.** A set $K \subset \mathbb{R}$ is compact if and only if K is closed and bounded.

Proposition 4.4.6. If $K \subset \mathbb{R}$ is compact and $\{x_n\}_{n\in I}$ is a sequence with $x_n \in K$ for every $n \in I$, then $\{x_n\}_{n\in I}$ has a convergent subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ with

$$\lim_{k\to\infty} x_{n_k} \in K. \tag{4.4.20}$$

Proof. Since K is bounded, $\{x_n\}_{n\in I}$ has a convergent subsequence $\{x_{n_k}\}_{k=1}^{\infty}$. Since K is closed, we must have $\lim_{k\to\infty} x_{n_k} \in K$. **Q.E.D.**

Proposition 4.4.7. Suppose $K \subset \mathbb{R}$ is such that whenever $\{x_n\}_{n\in I}$ is a sequence with $x_n \in K$ for every $n \in I$, then $\{x_n\}_{n\in I}$ has a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ with $\lim_{k\to\infty} x_{n_k} \in K$. Then K is compact.

Proof. Suppose K is unbounded. Then we may construct a sequence $\{x_n\}_{n=1}^{\infty}$ such that $x_n \in K$ and $|x_n| > n$ for $n = 1, 2, 3, \ldots$. Hence the only possible subsequential limits of $\{x_n\}_{n=1}^{\infty}$ would be $-\infty$ and $+\infty$, contradicting our assumptions. Thus K must be bounded.

Now suppose $\{x_n\}_{n\in I}$ is a convergent sequence with $x_n \in K$ for all $n \in I$. If $L = \lim_{n\to\infty} x_n$, then L is the only subsequential limit of $\{x_n\}_{n\in I}$. Hence, by the assumptions of the proposition, $L \in K$. Hence K is closed.

Since K is both closed and bounded, it is compact. **Q.E.D.**

Exercise 4.4.5. Show that a set $K \subset \mathbb{R}$ is compact if and only if every infinite subset of K has a limit point in K.

Exercise 4.4.6. Show that if K is compact, then $\sup K \in K$ and $\inf K \in K$.

Theorem 4.4.8. Given a set $K \subset \mathbb{R}$, the following are equivalent:

1. Every open cover of K has a finite subcover.

2. Every sequence in K has a subsequential limit in K.

3. Every infinite subset of K has a limit point in K.

Exercise 4.4.7. Suppose $K_1, K_2, K_3, \ldots$ are nonempty compact sets with

$$K_{n+1} \subset K_n$$

for $n = 1, 2, 3, \ldots$. Show that

$$\bigcap_{n=1}^{\infty} K_n$$

is nonempty.

Exercise 4.4.8. We say a collection of sets $\{D_\alpha : \alpha \in A\}$ has the *finite intersection property* if for every finite set $B \subset A$,

$$\bigcap_{\alpha \in B} D_\alpha \neq \emptyset.$$

Show that a set $K \subset \mathbb{R}$ is compact if and only for any collection

$$\{E_\alpha : \alpha \in A, E_\alpha = C_\alpha \cap K \text{ where } C_\alpha \subset \mathbb{R} \text{ is closed}\}$$

which has the finite intersection property we have

$$\bigcap_{\alpha \in A} E_\alpha \neq \emptyset.$$

Chapter 5

Limits and Continuity

5.1 Limits

Let $A \subset \mathbb{R}$ and let x be a limit point of A. In the following, we will let $S(A, x)$ denote the set of all convergent sequences $\{x_n\}_{n \in I}$ such that $x_n \in A$ for all $n \in I$, $x_n \neq x$ for all $n \in I$, and $\lim_{n \to \infty} x_n = x$. We will let $S^+(A, x)$ be the subset of $S(A, x)$ of sequences $\{x_n\}_{n \in I}$ for which $x_n > x$ for all $n \in I$ and $S^-(A, x)$ be the subset of $S(A, x)$ of sequences $\{x_n\}_{n \in I}$ for which $x_n < x$ for all $n \in I$.

Definition 5.1.1. Let $D \subset \mathbb{R}$, $f : D \to \mathbb{R}$, $L \in \mathbb{R}$, and suppose a is a limit point of D. We say the *limit* of f as x approaches a is L, denoted

$$\lim_{x \to a} f(x) = L, \tag{5.1.1}$$

if for every sequence $\{x_n\}_{n \in I} \in S(D, a)$,

$$\lim_{n \to \infty} f(x_n) = L. \tag{5.1.2}$$

If $S^+(D, a) \neq \emptyset$, we say the *limit from the right* of f as x approaches a is L, denoted

$$\lim_{x \to a^+} f(x) = L, \tag{5.1.3}$$

if for every sequence $\{x_n\}_{n \in I} \in S^+(D, a)$,

$$\lim_{n \to \infty} f(x_n) = L, \tag{5.1.4}$$

and, if $S^-(D, a) \neq \emptyset$, we say the *limit from the left* of f as x approaches a is L, denoted

$$\lim_{x \to a^-} f(x) = L, \tag{5.1.5}$$

if for every sequence $\{x_n\}_{n \in I} \in S^-(D, a)$,

$$\lim_{n \to \infty} f(x_n) = L. \tag{5.1.6}$$

We may also denote
$$\lim_{x\to a} f(x) = L \tag{5.1.7}$$
by writing
$$f(x) \to L \text{ as } x \to a. \tag{5.1.8}$$
Similarly, we may denote
$$\lim_{x\to a^+} f(x) = L \tag{5.1.9}$$
by writing
$$f(x) \to L \text{ as } x \downarrow a \tag{5.1.10}$$
and
$$\lim_{x\to a^-} f(x) = L \tag{5.1.11}$$
by writing
$$f(x) \to L \text{ as } x \uparrow a \tag{5.1.12}$$
We also let
$$f(a+) = \lim_{x\to a^+} f(x) \tag{5.1.13}$$
and
$$f(a-) = \lim_{x\to a^-} f(x). \tag{5.1.14}$$

It should be clear that if $\lim_{x\to a} f(x) = L$ and $S^+(D, a) \neq \emptyset$, then $f(a+) = L$. Similarly, if $\lim_{x\to a} f(x) = L$ and $S^-(D, a) \neq \emptyset$, then $f(a-) = L$.

Proposition 5.1.1. Suppose $D \subset \mathbb{R}$, $f : D \to \mathbb{R}$, and a is a limit point of D. If $f(a-) = f(a+) = L$, then $\lim_{x\to a} f(x) = L$.

Proof. Suppose $\{x_n\}_{n=m}^{\infty} \in S(D, a)$. Let
$$J^- = \{n : n \in \mathbb{Z}, x_n < a\} \tag{5.1.15}$$
and
$$J^+ = \{n : n \in \mathbb{Z}, x_n > a\}. \tag{5.1.16}$$
Suppose J^- is empty or finite and let $k = m - 1$ if $J^- = \emptyset$ and, otherwise, let k be the largest integer in J^-. Then $\{x_n\}_{n=k+1}^{\infty} \in S^+(D, a)$, and so
$$\lim_{n\to\infty} f(x_n) = f(a+) = L. \tag{5.1.17}$$
A similar argument shows that if J^+ is empty or finite, then
$$\lim_{n\to\infty} f(x_n) = f(a-) = L. \tag{5.1.18}$$

If neither J^- nor J^+ is finite or empty, then $\{x_n\}_{n\in J^-}$ and $\{x_n\}_{n\in J^+}$ are subsequences of $\{x_n\}_{n=m}^{\infty}$ with $\{x_n\}_{n\in J^-} \in S^-(D, a)$ and $\{x_n\}_{n\in J^+} \in S^+(D, a)$. Hence, given any $\epsilon > 0$, we may find integers N and M such that
$$|f(x_n) - L| < \epsilon \tag{5.1.19}$$

whenever $n \in \{j : j \in J^-, j > N\}$ and

$$|f(x_n) - L| < \epsilon \tag{5.1.20}$$

whenever $n \in \{j : j \in J^+, j > M\}$. Let P be the larger of N and M. Since $J^- \cup J^+ = \{j : j \in \mathbb{Z}^+, j \geq m\}$, it follows that

$$|f(x_n) - L| < \epsilon \tag{5.1.21}$$

whenever $n > P$. Hence $\lim\limits_{n\to\infty} f(x_n) = L$, and so $\lim\limits_{x\to a} f(x) = L$. **Q.E.D.**

Proposition 5.1.2. Suppose $D \subset \mathbb{R}$, a is a limit point of D, and $f : D \to \mathbb{R}$. If $\lim\limits_{x\to a} f(x) = L$ and $\alpha \in \mathbb{R}$, then

$$\lim_{x\to a} \alpha f(x) = \alpha L. \tag{5.1.22}$$

Proof. Suppose $\{x_n\}_{n\in I} \in S(D, a)$. Then

$$\lim_{n\to\infty} \alpha f(x_n) = \alpha \lim_{n\to\infty} f(x_n) = \alpha L. \tag{5.1.23}$$

Hence $\lim\limits_{x\to a} \alpha f(x) = \alpha L$. **Q.E.D.**

Proposition 5.1.3. Suppose $D \subset \mathbb{R}$, a is a limit point of D, $f : D \to \mathbb{R}$, and $g : D \to \mathbb{R}$. If $\lim\limits_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$, then

$$\lim_{x\to a} (f(x) + g(x)) = L + M. \tag{5.1.24}$$

Proof. Suppose $\{x_n\}_{n\in I} \in S(D, a)$. Then

$$\lim_{n\to\infty} (f(x_n) + g(x_n)) = \lim_{n\to\infty} f(x_n) + \lim_{n\to\infty} g(x_n) = L + M. \tag{5.1.25}$$

Hence $\lim\limits_{x\to a} (f(x) + g(x)) = L + M$. **Q.E.D.**

Proposition 5.1.4. Suppose $D \subset \mathbb{R}$, a is a limit point of D, $f : D \to \mathbb{R}$, and $g : D \to \mathbb{R}$. If $\lim\limits_{x\to a} f(x) = L$ and $\lim\limits_{x\to a} g(x) = M$, then

$$\lim_{x\to a} f(x)g(x) = LM. \tag{5.1.26}$$

Exercise 5.1.1. Prove the previous proposition.

Proposition 5.1.5. Suppose $D \subset \mathbb{R}$, a is a limit point of D, $f : D \to \mathbb{R}$, $g : D \to \mathbb{R}$, and $g(x) \neq 0$ for all $x \in D$. If $\lim\limits_{x\to a} f(x) = L$, $\lim\limits_{x\to a} g(x) = M$, and $M \neq 0$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L}{M}. \tag{5.1.27}$$

Exercise 5.1.2. Prove the previous proposition.

Proposition 5.1.6. Suppose $D \subset \mathbb{R}$, a is a limit point of D, $f : D \to \mathbb{R}$, and $f(x) \geq 0$ for all $x \in D$. If $\lim_{x \to a} f(x) = L$, then

$$\lim_{x \to a} \sqrt{f(x)} = \sqrt{L}. \tag{5.1.28}$$

Exercise 5.1.3. Prove the previous proposition.

Given $D \subset \mathbb{R}$, $f : D \to \mathbb{R}$, and $A \subset D$, we let

$$f(A) = \{y : y = f(x) \text{ for some } x \in A\}. \tag{5.1.29}$$

In particular, $f(D)$ denotes the range of f.

Proposition 5.1.7. Suppose $D \subset \mathbb{R}$, $E \subset \mathbb{R}$, a is a limit point of D, $g : D \to \mathbb{R}$, $f : E \to \mathbb{R}$, and $g(D) \subset E$. Moreover, suppose $\lim_{x \to a} g(x) = b$ and, for some $\epsilon > 0$, $g(x) \neq b$ for all $x \in (a - \epsilon, a + \epsilon) \cap D$. If $\lim_{x \to b} f(x) = L$, then

$$\lim_{x \to a} f \circ g(x) = L. \tag{5.1.30}$$

Proof. Suppose $\{x_n\}_{n \in I} \in S(D, a)$. Then

$$\lim_{n \to \infty} g(x_n) = b. \tag{5.1.31}$$

Let $N \in \mathbb{Z}^+$ such that $|x_n - a| < \epsilon$ whenever $n > N$. Then

$$\{g(x_n)\}_{n=N+1}^{\infty} \in S(E, b), \tag{5.1.32}$$

so

$$\lim_{n \to \infty} f(g(x_n)) = L. \tag{5.1.33}$$

Thus $\lim_{x \to a} f \circ g(x) = L$. **Q.E.D.**

Example 5.1.1. Let

$$g(x) = \begin{cases} 0, & \text{if } x \neq 0, \\ 1, & \text{if } x = 0. \end{cases}$$

If $f(x) = g(x)$, then

$$f \circ g(x) = \begin{cases} 1, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

Hence $\lim_{x \to 0} f \circ g(x) = 1$, although $\lim_{x \to 0} g(x) = 0$ and $\lim_{x \to 0} f(x) = 0$.

5.1.1 Limits of polynomials and rational functions

Example 5.1.2. If $c \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ is given by $f(x) = c$ for all $x \in \mathbb{R}$, then clearly $\lim_{x \to a} f(x) = c$ for any $a \in \mathbb{R}$.

Example 5.1.3. Suppose $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = x$ for all $x \in \mathbb{R}$. If, for any $a \in \mathbb{R}$, $\{x_n\}_{n \in I} \in S(\mathbb{R}, a)$, then

$$\lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} x_n = a.$$

Hence $\lim_{x \to a} x = a$.

Example 5.1.4. Suppose $n \in \mathbb{Z}^+$ and $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = x^n$. Then

$$\lim_{x \to a} f(x) = \lim_{x \to a} x^n = \prod_{i=1}^{n} \lim_{x \to a} x = a^n.$$

Definition 5.1.2. If $n \in \mathbb{Z}$, $n \geq 0$, and $b_0, b_1, \ldots, b_n$ are real numbers with $b_n \neq 0$, then we call the function $p : \mathbb{R} \to \mathbb{R}$ defined by

$$p(x) = b_n x^n + b_{n-1} x^{n-1} + \cdots + b_1 x + b_0 \tag{5.1.34}$$

a *polynomial* of *degree n*.

Exercise 5.1.4. Show that if f is a polynomial and $a \in \mathbb{R}$, then $\lim_{x \to a} f(x) = f(a)$.

Definition 5.1.3. Suppose p and q are polynomials and

$$D = \{x : x \in \mathbb{R}, q(x) \neq 0\}. \tag{5.1.35}$$

We call the function $r : D \to \mathbb{R}$ defined by

$$r(x) = \frac{p(x)}{q(x)} \tag{5.1.36}$$

a *rational function*.

Exercise 5.1.5. Show that if f is a rational function and a is in the domain of f, then $\lim_{x \to a} f(x) = f(a)$.

Exercise 5.1.6. Suppose $D \subset \mathbb{R}$, $a \in D$ is a limit point of D, and $\lim_{x \to a} f(x) = L$. If $E = D \setminus \{a\}$ and $g : E \to \mathbb{R}$ is defined by $g(x) = f(x)$ for all $x \in E$, show that $\lim_{x \to a} g(x) = L$.

Exercise 5.1.7. Evaluate

$$\lim_{x \to 1} \frac{x^5 - 1}{x^3 - 1}.$$

Exercise 5.1.8. Suppose $D \subset \mathbb{R}$, a is a limit point of D, $f : D \to \mathbb{R}$, $g : D \to \mathbb{R}$, $h : D \to \mathbb{R}$, and $f(x) \leq h(x) \leq g(x)$ for all $x \in D$. If $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = L$, show that $\lim_{x \to a} h(x) = L$. (This is the *squeeze theorem* for limits of functions.)

Note that the above results which have been stated for limits will hold as well for the appropriate one-sided limits, that is, limits from the right or from the left.

Exercise 5.1.9. Suppose

$$f(x) = \begin{cases} x + 1, & \text{if } x < 0, \\ 4, & \text{if } x = 0, \\ x^2, & \text{if } x > 0. \end{cases}$$

Evaluate $f(0)$, $f(0-)$, and $f(0+)$. Does $\lim_{x \to 0} f(x)$ exist?

5.1.2 Equivalent definitions

Proposition 5.1.8. Suppose $D \subset \mathbb{R}$, a is a limit point of D, and $f : D \to \mathbb{R}$. Then $\lim_{x \to a} f(x) = L$ if and only if for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|f(x) - L| < \epsilon \text{ whenever } x \neq a \text{ and } x \in (a - \delta, a + \delta) \cap D. \tag{5.1.37}$$

Proof. Suppose $\lim_{x \to a} f(x) = L$. Suppose there exists an $\epsilon > 0$ such that for every $\delta > 0$ there exists $x \in (a - \delta, a + \delta) \cap D$, $x \neq a$, for which $|f(x) - L| \geq \epsilon$. For $n = 1, 2, 3, \ldots$, choose

$$x_n \in \left(a - \frac{1}{n}, a + \frac{1}{n}\right) \cap D, \tag{5.1.38}$$

$x_n \neq a$, such that $|f(x_n) - L| \geq \epsilon$. Then $\{x_n\}_{n=1}^{\infty} \in S(D, a)$, but $\{f(x_n)\}_{n=1}^{\infty}$ does not converge to L, contradicting the assumption that $\lim_{x \to a} f(x) = L$.

Now suppose that for every $\epsilon > 0$ there exists $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $x \neq a$ and $x \in (a - \delta, a + \delta) \cap D$. Let $\{x_n\}_{n \in I} \in S(D, a)$. Given $\epsilon > 0$, let $\delta > 0$ be such that $|f(x) - L| < \epsilon$ whenever $x \neq a$ and $x \in (a - \delta, a + \delta) \cap D$. Choose $N \in \mathbb{Z}$ such that $|x_n - a| < \delta$ whenever $n > N$. Then $|f(x_n) - L| < \epsilon$ for all $n > N$. Hence $\lim_{n \to \infty} f(x_n) = L$, and so $\lim_{x \to a} f(x) = L$. **Q.E.D.**

The proofs of the next two propositions are analogous.

Proposition 5.1.9. Suppose $D \subset \mathbb{R}$, a is a limit point of D, $f : D \to \mathbb{R}$, and $S^-(D, a) \neq \emptyset$. Then $\lim_{x \to a^-} f(x) = L$ if and only if for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|f(x) - L| < \epsilon \text{ whenever } x \in (a - \delta, a) \cap D. \tag{5.1.39}$$

Proposition 5.1.10. Suppose $D \subset \mathbb{R}$, a is a limit point of D, $f : D \to \mathbb{R}$, and $S^+(D, a) \neq \emptyset$. Then $\lim_{x \to a^+} f(x) = L$ if and only if for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|f(x) - L| < \epsilon \text{ whenever } x \in (a, a + \delta) \cap D. \tag{5.1.40}$$

5.1.3 Examples

Example 5.1.5. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational,} \\ 0, & \text{if } x \text{ is irrational.} \end{cases}$$

Let $a \in \mathbb{R}$. Since every open interval contains both rational and irrational numbers, for any $\delta > 0$ and any choice of $L \in \mathbb{R}$, there will exist $x \in (a - \delta, a + \delta)$, $x \neq a$, such that

$$|f(x) - L| \geq \frac{1}{2}.$$

Hence $\lim_{x \to a} f(x)$ does not exist for any real number a.

Example 5.1.6. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x, & \text{if } x \text{ is rational,} \\ 0, & \text{if } x \text{ is irrational.} \end{cases}$$

Then $\lim_{x \to 0} f(x) = 0$ since, given $\epsilon > 0$, $|f(x)| < \epsilon$ provided $|x| < \epsilon$.

Exercise 5.1.10. Show that if f is as given in the previous example and $a \neq 0$, then $\lim_{x \to a} f(x)$ does not exist.

Exercise 5.1.11. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} \dfrac{1}{q}, & \text{if } x \text{ is rational and } x = \dfrac{p}{q}, \\ 0, & \text{if } x \text{ is irrational,} \end{cases}$$

where p and q are taken to be relatively prime integers with $q > 0$, and we take $q = 1$ when $x = 0$. Show that, for any real number a, $\lim_{x \to a} f(x) = 0$.

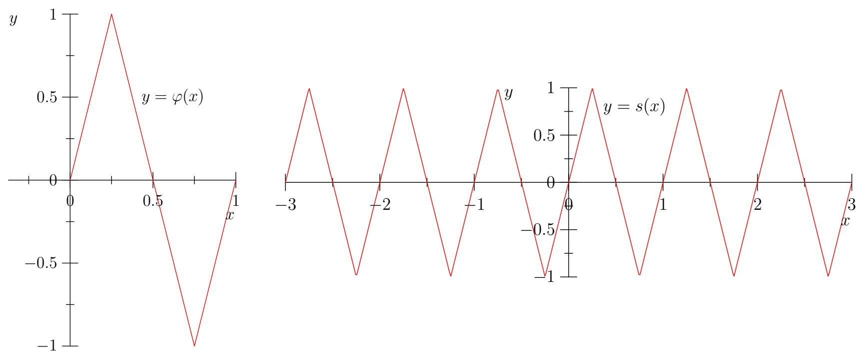

Figure 5.1.1: Graphs of $y = \varphi(x)$ and $y = s(x)$

Example 5.1.7. Define $\varphi : [0, 1] \to [-1, 1]$ by

$$\varphi(x) = \begin{cases} 4x, & \text{if } 0 \le x \le \frac{1}{4}, \\ 2 - 4x, & \text{if } \frac{1}{4} < x < \frac{3}{4}, \\ 4x - 4, & \text{if } \frac{3}{4} \le x \le 1. \end{cases}$$

Next define $s : \mathbb{R} \to \mathbb{R}$ by $s(x) = \varphi(x - \lfloor x \rfloor)$, where $\lfloor x \rfloor$ denotes the largest integer less than or equal to x (that is, $\lfloor x \rfloor$ is the *floor* of x). The function s is an example of a *sawtooth* function. See the graphs of φ and s in Figure 5.1.1. Note that for any $n \in \mathbb{Z}$,

$$s([n, n + 1]) = [-1, 1].$$

Now let $D = \mathbb{R} \setminus \{0\}$ and define $\sigma : D \to \mathbb{R}$ by

$$\sigma(x) = s\left(\frac{1}{x}\right).$$

See the graph of σ in Figure 5.1.2. Note that for any $n \in \mathbb{Z}^+$,

$$\sigma\left(\left[\frac{1}{n+1}, \frac{1}{n}\right]\right) = s([n, n + 1]) = [-1, 1].$$

Hence for any $\epsilon > 0$, $\sigma((0, \epsilon)) = [-1, 1]$, and so $\lim_{x \to 0^+} \sigma(x)$ does not exist. Similarly, neither $\lim_{x \to 0^-} \sigma(x)$ nor $\lim_{x \to 0} \sigma(x)$ exist.

Example 5.1.8. Let s be the sawtooth function of the previous example and let $D = \mathbb{R} \setminus \{0\}$. Define $\psi : D \to \mathbb{R}$ by

$$\psi(x) = xs\left(\frac{1}{x}\right).$$

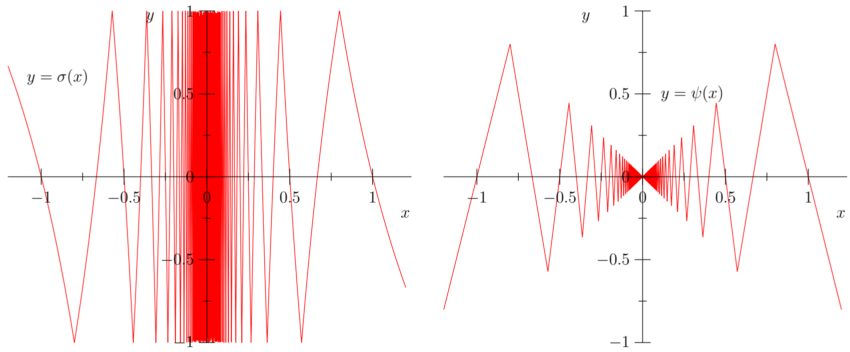

Figure 5.1.2: Graphs of $y = \sigma(x)$ and $y = \psi(x)$

See Figure 5.1.2 for the graph of ψ. Then for all $x \in D$,

$$-|x| \leq \psi(x) \leq |x|,$$

and so $\lim\limits_{x \to 0} \psi(x) = 0$ by the squeeze theorem.

Definition 5.1.4. Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$. We say f is *bounded* if there exists a real number B such that $|f(x)| \leq B$ for all $x \in D$.

Exercise 5.1.12. Suppose $f : \mathbb{R} \to \mathbb{R}$ is bounded. Show that $\lim\limits_{x \to 0} x f(x) = 0$.

5.2 Monotonic functions

Definition 5.2.1. Suppose $D \subset \mathbb{R}$, $f : D \to \mathbb{R}$, and $(a, b) \subset D$. We say f is *increasing* on (a, b) if $f(x) < f(y)$ whenever $a < x < y < b$; we say f is *decreasing* on (a, b) if $f(x) > f(y)$ whenever $a < x < y < b$; we say f is *nondecreasing* on (a, b) if $f(x) \leq f(y)$ whenever $a < x < y < b$; and we say f is *nonincreasing* on (a, b) if $f(x) \geq f(y)$ whenever $a < x < y < b$. We will say f is *monotonic* on (a, b) if f is either nondecreasing or nonincreasing on (a, b) and we will say f is *strictly monotonic* on (a, b) if f is either increasing or decreasing on (a, b).

Proposition 5.2.1. If f is monotonic on (a, b), then $f(c+)$ and $f(c-)$ exist for every $c \in (a, b)$.

Proof. Suppose f is nondecreasing on (a, b). Let $c \in (a, b)$ and let

$$\lambda = \sup\{f(x) : a < x < c\}. \tag{5.2.1}$$

Note that $\lambda \leq f(c) < +\infty$. Given any $\epsilon > 0$, there must exist $\delta > 0$ such that

$$\lambda - \epsilon < f(c - \delta) \leq \lambda. \tag{5.2.2}$$

Since f is nondecreasing, it follows that

$$|f(x) - \lambda| < \epsilon \tag{5.2.3}$$

whenever $x \in (c - \delta, c)$. Thus $f(c-) = \lambda$. A similar argument shows that $f(c+) = \kappa$ where

$$\kappa = \inf\{f(x) : c < x < b\}. \tag{5.2.4}$$

If f is nonincreasing, similar arguments yield

$$f(c-) = \inf\{f(x) : a < x < c\} \tag{5.2.5}$$

and

$$f(c+) = \sup\{f(x) : c < x < b\}. \tag{5.2.6}$$

Q.E.D.

Proposition 5.2.2. If f is nondecreasing on (a, b) and $a < x < y < b$, then

$$f(x+) \leq f(y-). \tag{5.2.7}$$

Proof. By the previous proposition,

$$f(x+) = \inf\{f(t) : x < t < b\} \tag{5.2.8}$$

and

$$f(y-) = \sup\{f(t) : a < t < y\}. \tag{5.2.9}$$

Since f is nondecreasing,

$$\inf\{f(t) : x < t < b\} = \inf\{f(t) : x < t < y\} \tag{5.2.10}$$

and

$$\sup\{f(t) : a < t < y\} = \sup\{f(t) : x < t < y\}. \tag{5.2.11}$$

Thus

$$f(x+) = \inf\{f(t) : x < t < y\} \leq \sup\{f(t) : x < t < y\} = f(y-). \tag{5.2.12}$$

Q.E.D.

Exercise 5.2.1. Let $\varphi : \mathbb{Q} \cap [0, 1] \to \mathbb{Z}^+$ be a one-to-one correspondence. Define $f : [0, 1] \to \mathbb{R}$ by

$$f(x) = \sum_{\substack{q \in \mathbb{Q} \cap [0,1] \\ q \leq x}} \frac{1}{2^{\varphi(q)}}.$$

a. Show that f is increasing on $(0, 1)$.

b. Show that for any $x \in \mathbb{Q} \cap (0, 1)$, $f(x-) < f(x)$ and $f(x+) = f(x)$.

c. Show that for any irrational a, $0 < a < 1$, $\lim_{x \to a} f(x) = f(a)$.

5.3 Limits to infinity and infinite limits

Definition 5.3.1. Let $D \subset \mathbb{R}$, $f : D \to \mathbb{R}$, and suppose a is a limit point of D. We say that f *diverges* to $+\infty$ as x approaches a, denoted

$$\lim_{x \to a} f(x) = +\infty, \tag{5.3.1}$$

if for every real number M there exists a $\delta > 0$ such that

$$f(x) > M \text{ whenever } x \neq a \text{ and } x \in (a - \delta, a + \delta) \cap D. \tag{5.3.2}$$

Similarly, we say that that f *diverges* to $-\infty$ as x approaches a, denoted

$$\lim_{x \to a} f(x) = -\infty, \tag{5.3.3}$$

if for every real number M there exists a $\delta > 0$ such that

$$f(x) < M \text{ whenever } x \neq a \text{ and } x \in (a - \delta, a + \delta) \cap D. \tag{5.3.4}$$

Exercise 5.3.1. Provide definitions for

a. $\displaystyle\lim_{x \to a^+} f(x) = +\infty$,

b. $\displaystyle\lim_{x \to a^-} f(x) = +\infty$,

c. $\displaystyle\lim_{x \to a^+} f(x) = -\infty$,

d. $\displaystyle\lim_{x \to a^-} f(x) = -\infty$.

Model your definitions on the preceding definitions.

Exercise 5.3.2. Show that $\displaystyle\lim_{x \to 4^+} \frac{7}{4 - x} = -\infty$ and $\displaystyle\lim_{x \to 4^-} \frac{7}{4 - x} = +\infty$.

Definition 5.3.2. Suppose $D \subset \mathbb{R}$ does not have an upper bound, $f : D \to \mathbb{R}$, and $L \in \mathbb{R}$. We say that the *limit* of f as x approaches $+\infty$ is L, denoted

$$\lim_{x \to +\infty} f(x) = L, \tag{5.3.5}$$

if for every $\epsilon > 0$ there exists a real number M such that

$$|f(x) - L| < \epsilon \text{ whenever } x \in (M, +\infty) \cap D. \tag{5.3.6}$$

Definition 5.3.3. Suppose $D \subset \mathbb{R}$ does not have an lower bound, $f : D \to \mathbb{R}$, and $L \in \mathbb{R}$. We say that the *limit* of f as x approaches $-\infty$ is L, denoted

$$\lim_{x \to -\infty} f(x) = L, \tag{5.3.7}$$

if for every $\epsilon > 0$ there exists a real number M such that

$$|f(x) - L| < \epsilon \text{ whenever } x \in (-\infty, M) \cap D. \tag{5.3.8}$$

Exercise 5.3.3. Verify that $\lim_{x \to +\infty} \frac{x+1}{x+2} = 1$.

Exercise 5.3.4. Provide definitions for

a. $\lim_{x \to +\infty} f(x) = +\infty$,

b. $\lim_{x \to +\infty} f(x) = -\infty$,

c. $\lim_{x \to -\infty} f(x) = +\infty$,

d. $\lim_{x \to -\infty} f(x) = -\infty$.

Model your definitions on the preceding definitions.

Exercise 5.3.5. Suppose

$$f(x) = ax^3 + bx^2 + cx + d,$$

where $a, b, c, d \in \mathbb{R}$ and $a > 0$. Show that

$$\lim_{x \to +\infty} f(x) = +\infty \text{ and } \lim_{x \to -\infty} f(x) = -\infty.$$

5.4 Continuous Functions

5.4.1 Continuity at a point

Definition 5.4.1. Suppose $D \subset \mathbb{R}$, $f : D \to \mathbb{R}$, and $a \in D$. We say f is *continuous* at a if either a is an isolated point of D or $\lim_{x \to a} f(x) = f(a)$. If f is not continuous at a, we say f is *discontinuous* at a, or that f has a *discontinuity* at a.

Example 5.4.1. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational,} \\ 0, & \text{if } x \text{ is irrational.} \end{cases}$$

Then, by Example 5.1.5, f is discontinuous at every $x \in \mathbb{R}$.

Example 5.4.2. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x, & \text{if } x \text{ is rational,} \\ 0, & \text{if } x \text{ is irrational.} \end{cases}$$

Then, by Example 5.1.6 and Exercise 5.1.10, f is continuous at 0, but discontinuous at every $x \neq 0$.

If $D \subset \mathbb{R}$, $\alpha \in \mathbb{R}$, $f : D \to \mathbb{R}$, and $g : D \to \mathbb{R}$, then we define $\alpha f : D \to \mathbb{R}$ by

$$(\alpha f)(x) = \alpha f(x), \tag{5.4.1}$$

$f + g : D \to \mathbb{R}$ by

$$(f + g)(x) = f(x) + g(x), \tag{5.4.2}$$

and $fg : D \to \mathbb{R}$ by

$$(fg)(x) = f(x)g(x). \tag{5.4.3}$$

Moreover, if $g(x) \neq 0$ for all $x \in D$, we define $\frac{f}{g} : D \to \mathbb{R}$ by

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}. \tag{5.4.4}$$

Proposition 5.4.1. Suppose $D \subset \mathbb{R}$, $\alpha \in \mathbb{R}$, $f : D \to \mathbb{R}$, and $g : D \to \mathbb{R}$. If f and g are continuous at a, then αf, $f + g$, and fg are all continuous at a. Moreover, if $g(x) \neq 0$ for all $x \in D$, then $\frac{f}{g}$ is continuous at a.

Exercise 5.4.1. Prove the previous proposition.

Proposition 5.4.2. Suppose $D \subset \mathbb{R}$, $f : D \to \mathbb{R}$, $f(x) \geq 0$ for all $x \in D$, and f is continuous at $a \in D$. If $g : D \to \mathbb{R}$ is defined by $g(x) = \sqrt{f(x)}$, then g is continuous at a.

Exercise 5.4.2. Prove the previous proposition.

Proposition 5.4.3. Suppose $D \subset \mathbb{R}$, $f : D \to \mathbb{R}$, and $a \in D$. Then f is continuous at a if and only if for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$|f(x) - f(a)| < \epsilon \text{ whenever } x \in (a - \delta, a + \delta) \cap D. \tag{5.4.5}$$

Proof. Suppose f is continuous at a. If a is an isolated point of D, then there exists a $\delta > 0$ such that

$$(a - \delta, a + \delta) \cap D = \{a\}. \tag{5.4.6}$$

Then for any $\epsilon > 0$, if $x \in (a - \delta, a + \delta) \cap D$, then $x = a$, and so

$$|f(x) - f(a)| = |f(a) - f(a)| = 0 < \epsilon. \tag{5.4.7}$$

If a is a limit point of D, then $\lim_{x \to a} f(x) = f(a)$ implies that for any $\epsilon > 0$ there exists $\delta > 0$ such that

$$|f(x) - f(a)| < \epsilon \text{ whenever } x \in (a - \delta, a + \delta) \cap D. \tag{5.4.8}$$

Now suppose that for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$|f(x) - f(a)| < \epsilon \text{ whenever } x \in (a - \delta, a + \delta) \cap D. \tag{5.4.9}$$

If a is an isolated point, then f is continuous at a. If a is a limit point, then this condition implies $\lim_{x \to a} f(x) = f(a)$, and so f is continuous at a. **Q.E.D.**

From the preceding, it should be clear that a function $f : D \to \mathbb{R}$ is continuous at a point a of D if and only if for every sequence $\{x_n\}_{n \in I}$ with $x_n \in D$ for every $n \in I$ and $\lim_{n \to \infty} x_n = a$, $\lim_{n \to \infty} f(x_n) = f(a)$.

Exercise 5.4.3. Show that if $f : D \to \mathbb{R}$ is continuous at $a \in D$ and $f(a) > 0$, then there exists an open interval I such that $a \in I$ and $f(x) > 0$ for every $x \in I \cap D$.

Proposition 5.4.4. Suppose $D \subset \mathbb{R}$, $E \subset \mathbb{R}$, $g : D \to \mathbb{R}$, $f : E \to \mathbb{R}$, $g(D) \subset E$, and $a \in D$. If g is continuous at a and f is continuous at $g(a)$, then $f \circ g$ is continuous at a.

Proof. Let $\{x_n\}_{n \in I}$ be a sequence with $x_n \in D$ for every $n \in I$ and $\lim_{n \to \infty} x_n = a$. Then, since g is continuous at a, $\{g(x_n)\}_{n \in I}$ is a sequence with $g(x_n) \in E$ for every $n \in I$ and $\lim_{n \to \infty} g(x_n) = g(a)$. Hence, since f is continuous at $g(a)$, $\lim_{n \to \infty} f(g(x_n)) = f(g(a))$. That is,

$$\lim_{n \to \infty} (f \circ g)(x_n) = (f \circ g)(a). \tag{5.4.10}$$

Hence $f \circ g$ is continuous at a. **Q.E.D.**

Definition 5.4.2. Let $D \subset \mathbb{R}$, $f : D \to \mathbb{R}$, and $a \in D$. If f is not continuous at a but both $f(a-)$ and $f(a+)$ exist, then we say f has a *simple discontinuity* at a.

Proposition 5.4.5. Suppose f is monotonic on the interval (a, b). Then every discontinuity of f in (a, b) is a simple discontinuity. Moreover, if E is the set of points in (a, b) at which f is discontinuous, then either $E = \emptyset$, E is finite, or E is countable.

Proof. The first statement follows immediately from Proposition 5.2.1. For the second statement, suppose f is nondecreasing and suppose E is nonempty. From Exercise 2.1.26 and the the proof of Proposition 5.2.1, it follows that for every $x \in (a, b)$,

$$f(x-) \leq f(x) \leq f(x+). \tag{5.4.11}$$

Hence $x \in E$ if and only if $f(x-) < f(x+)$. Hence for every $x \in E$, we may choose a rational number r_x such that $f(x-) < r_x < f(x+)$. Now if $x, y \in E$ with $x < y$, then, by Proposition 5.2.2,

$$r_x < f(x+) \leq f(y-) < r_y, \tag{5.4.12}$$

so $r_x \neq r_y$. Thus we have a one-to-one correspondence between E and a subset of $\mathbb{Q}$, and so E is either finite or countable. A similar argument holds if f is nonincreasing. **Q.E.D.**

Exercise 5.4.4. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} \frac{1}{q}, & \text{if } x \text{ is rational and } x = \frac{p}{q}, \\ 0, & \text{if } x \text{ is irrational,} \end{cases}$$

where p and q are taken to be relatively prime integers with $q > 0$, and we take $q = 1$ when $x = 0$. Show that f is continuous at every irrational number and has a simple discontinuity at every rational number.

5.4.2 Continuity on a set

Definition 5.4.3. Suppose $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$. We say f is *continuous on* D if f is continuous at every point $a \in D$.

Proposition 5.4.6. If f is a polynomial, then f is continuous on $\mathbb{R}$.

Proposition 5.4.7. If $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ is a rational function, then f is continuous on D.

Exercise 5.4.5. Explain why the function $f(x) = \sqrt{1 - x^2}$ is continuous on $[-1, 1]$.

Exercise 5.4.6. Discuss the continuity of the function

$$f(x) = \begin{cases} x + 1, & \text{if } x < 0, \\ 4, & \text{if } x = 0, \\ x^2, & \text{if } x > 0. \end{cases}$$

If $D \subset \mathbb{R}$, $f : D \to \mathbb{R}$, and $E \subset \mathbb{R}$, we let

$$f^{-1}(E) = \{x : f(x) \in E\}. \tag{5.4.13}$$

Proposition 5.4.8. Suppose $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$. Then f is continuous on D if and only if for every open set $V \subset \mathbb{R}$, $f^{-1}(V) = U \cap D$ for some open set $U \subset \mathbb{R}$.

Proof. Suppose f is continuous on D and $V \subset \mathbb{R}$ is an open set. If $V \cap f(D) = \emptyset$, then $f^{-1}(V) = \emptyset$, which is open. So suppose $V \cap f(D) \neq \emptyset$ and let $a \in f^{-1}(V)$. Since V is open and $f(a) \in V$, there exists $\epsilon_a > 0$ such that

$$(f(a) - \epsilon_a, f(a) + \epsilon_a) \subset V. \tag{5.4.14}$$

Since f is continuous, there exists $\delta_a > 0$ such that

$$f((a - \delta_a, a + \delta_a) \cap D) \subset (f(a) - \epsilon_a, f(a) + \epsilon_a) \subset V. \tag{5.4.15}$$

That is, $(a - \delta_a, a + \delta_a) \cap D \subset f^{-1}(V)$. Let

$$U = \bigcup_{a \in f^{-1}(V)} (a - \delta_a, a + \delta_a). \tag{5.4.16}$$

Then U is open and $f^{-1}(V) = U \cap D$.

Now suppose that for every open set $V \subset \mathbb{R}$, $f^{-1}(V) = U \cap D$ for some open set $U \subset \mathbb{R}$. Let $a \in D$ and let $\epsilon > 0$ be given. Since $(f(a) - \epsilon, f(a) + \epsilon)$ is open, there exists an open set U such that

$$U \cap D = f^{-1}((f(a) - \epsilon, f(a) + \epsilon)). \tag{5.4.17}$$

Since U is open and $a \in U$, there exists $\delta > 0$ such that $(a - \delta, a + \delta) \subset U$. But then

$$f((a - \delta, a + \delta) \cap D) \subset (f(a) - \epsilon, f(a) + \epsilon). \tag{5.4.18}$$

That is, if $x \in (a - \delta, a + \delta) \cap D$, then $|f(x) - f(a)| < \epsilon$. Hence f is continuous at a. **Q.E.D.**

Exercise 5.4.7. Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$. For any $E \subset \mathbb{R}$, show that $f^{-1}(\mathbb{R} \setminus E) = (\mathbb{R} \setminus f^{-1}(E)) \cap D$.

Exercise 5.4.8. Let A be a set and, for each $\alpha \in A$, let $U_\alpha \subset \mathbb{R}$. Given $D \subset \mathbb{R}$ and a function $f : D \to \mathbb{R}$, show that

$$\bigcup_{\alpha \in A} f^{-1}(U_\alpha) = f^{-1}\left(\bigcup_{\alpha \in A} U_\alpha\right)$$

and

$$\bigcap_{\alpha \in A} f^{-1}(U_\alpha) = f^{-1}\left(\bigcap_{\alpha \in A} U_\alpha\right).$$

Exercise 5.4.9. Suppose $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$. Show that f is continuous on D if and only if for every closed set $C \subset \mathbb{R}$, $f^{-1}(C) = F \cap D$ for some closed set $F \subset \mathbb{R}$.

Exercise 5.4.10. Let $D \subset \mathbb{R}$. We say a function $f : D \to \mathbb{R}$ is *Lipschitz* if there exists $\alpha \in \mathbb{R}$, $\alpha > 0$, such that $|f(x) - f(y)| \leq \alpha |x - y|$ for all $x, y \in D$. Show that if f is Lipschitz, then f is continuous.

5.4.3 Intermediate Value Theorem

Theorem 5.4.9 (Intermediate Value Theorem)**.** Suppose $a, b \in \mathbb{R}$, $a < b$, and $f : [a, b] \to \mathbb{R}$. If f is continuous and $s \in \mathbb{R}$ is such that either $f(a) \le s \le f(b)$ or $f(b) \le s \le f(a)$, then there exists $c \in [a, b]$ such that $f(c) = s$.

Proof. Suppose $f(a) < f(b)$ and $f(a) < s < f(b)$. Let

$$c = \sup\{x : x \in [a, b], f(x) \le s\}. \tag{5.4.19}$$

Suppose $f(c) < s$. Then $c < b$ and, since f is continuous at c, there exists a $\delta > 0$ such that $f(x) < s$ for all $x \in (c, c + \delta)$. But then $f\left(c + \frac{\delta}{2}\right) < s$, contradicting the definition of c. Similarly, if $f(c) > s$, then $c > a$ and there exists $\delta > 0$ such that $f(x) > s$ for all $x \in (c - \delta, c)$, again contradicting the definition of c. Hence we must have $f(c) = s$. **Q.E.D.**

Example 5.4.3. Suppose $a \in \mathbb{R}$, $a > 0$, and consider $f(x) = x^n - a$ where $n \in \mathbb{Z}$, $n > 1$. Then $f(0) = -a < 0$ and

$$\begin{aligned} f(1 + a) &= (1 + a)^n - a \\ &= 1 + na + \sum_{i=2}^{n} \binom{n}{i} a^i - a \\ &= 1 + (n - 1)a + \sum_{i=2}^{n} \binom{n}{i} a^i > 0, \end{aligned}$$

where $\binom{n}{i}$ is the *binomial coefficient*

$$\binom{n}{i} = \frac{n!}{i!(n - i)!}. \tag{5.4.20}$$

Hence, by the Intermediate Value Theorem, there exists a real number $\gamma > 0$ such that $\gamma^n = a$. Moreover, there is only one such γ since f is increasing on $(0, +\infty)$.

We call γ the *nth root* of a, and write

$$\gamma = \sqrt[n]{a} \tag{5.4.21}$$

or

$$\gamma = a^{\frac{1}{n}}. \tag{5.4.22}$$

Moreover, if $a \in \mathbb{R}$, $a < 0$, $n \in Z^+$ is odd, and γ is the nth root of $-a$, then

$$(-\gamma)^n = (-1)^n(\gamma)^n = (-1)(-a) = a. \tag{5.4.23}$$

That is, $-\gamma$ is the nth root of a.

Definition 5.4.4. If $n = \frac{p}{q} \in \mathbb{Q}$ with $q \in \mathbb{Z}^+$, then we define

$$x^n = \left(\sqrt[q]{x}\right)^p \tag{5.4.24}$$

for all real $x \ge 0$.

Exercise 5.4.11. Explain why the equation $x^5 + 4x^2 - 16 = 0$ has a solution in the interval $(0, 2)$.

Exercise 5.4.12. Give an example of a closed interval $[a, b] \subset \mathbb{R}$ and a function $f : [a, b] \to \mathbb{R}$ which do not satisfy the conclusion of the Intermediate Value Theorem.

Exercise 5.4.13. Show that if $I \subset \mathbb{R}$ is an interval and $f : I \to \mathbb{R}$ is continuous, then $f(I)$ is an interval.

Exercise 5.4.14. Suppose $f : (a, b) \to \mathbb{R}$ is continuous and strictly monotonic. Let $(c, d) = f((a, b))$. Show that $f^{-1} : (c, d) \to (a, b)$ is strictly monotonic and continuous.

Exercise 5.4.15. Let $n \in \mathbb{Z}^+$. Show that the function $f(x) = \sqrt[n]{x}$ is continuous on $(0, +\infty)$.

Exercise 5.4.16. Use the method of bisection to give another proof of the Intermediate Value Theorem.

5.4.4 Extreme Value Theorem

Theorem 5.4.10. Suppose $D \subset \mathbb{R}$ is compact and $f : D \to \mathbb{R}$ is continuous. Then $f(D)$ is compact.

Proof. Given a sequence $\{y_n\}_{n\in I}$ in $f(D)$, choose a sequence $\{x_n\}_{n\in I}$ such that $f(x_n) = y_n$. Since D is compact, $\{x_n\}_{n\in I}$ has a convergent subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ with

$$\lim_{k\to\infty} x_{n_k} = x \in D. \tag{5.4.25}$$

Let $y = f(x)$. Then $y \in f(D)$ and, since f is continuous,

$$y = \lim_{k\to\infty} f(x_{n_k}) = \lim_{k\to\infty} y_{n_k}. \tag{5.4.26}$$

Hence $f(D)$ is compact. **Q.E.D.**

Exercise 5.4.17. Prove the previous theorem using the open cover definition of a compact set.

Theorem 5.4.11 (Extreme Value Theorem)**.** Suppose $D \subset \mathbb{R}$ is compact and $f : D \to \mathbb{R}$ is continuous. Then there exists $a \in D$ such that $f(a) \geq f(x)$ for all $x \in D$ and there exists $b \in D$ such that $f(b) \leq f(x)$ for all $x \in D$.

Proof. Let $s = \sup f(D)$ and $t = \inf f(D)$. Then $s \in f(D)$, so there exists $a \in D$ such that $f(a) = s$, and $t \in f(D)$, so there exists $b \in D$ such that $f(b) = t$. **Q.E.D.**

As a consequence of the Extreme Value Theorem, a continuous function on a closed bounded interval attains both a maximum and a minimum value.

Exercise 5.4.18. Find an example of a closed bounded interval $[a, b]$ and a function $f : [a, b] \to \mathbb{R}$ such that f attains neither a maximum nor a minimum value on $[a, b]$.

Exercise 5.4.19. Find an example of a bounded interval I and a function $f : I \to \mathbb{R}$ which is continuous on I such that f attains neither a maximum nor a minimum value on I.

Exercise 5.4.20. Suppose $K \subset \mathbb{R}$ is compact and $a \notin K$. Show that there exists $b \in K$ such that $|b - a| \leq |x - a|$ for all $x \in K$.

Proposition 5.4.12. Suppose $D \subset \mathbb{R}$ is compact, $f : D \to \mathbb{R}$ is continuous and one-to-one, and $E = f(D)$. Then $f^{-1} : E \to D$ is continuous.

Proof. Let $V \subset \mathbb{R}$ be an open set. We need to show that $f(V \cap D) = U \cap E$ for some open set $U \subset \mathbb{R}$. Let $C = D \cap (\mathbb{R} \setminus V)$. Then C is a closed subset of D, and so is compact. Hence $f(C)$ is a compact subset of E. Thus $f(C)$ is closed, and so $U = \mathbb{R} \setminus f(C)$ is open. Moreover, $U \cap E = E \setminus f(C) = f(V \cap D)$. Thus f^{-1} is continuous.

Exercise 5.4.21. Suppose $f : [0, 1] \cup (2, 3] \to [0, 2]$ by

$$f(x) = \begin{cases} x, & \text{if } 0 \leq x \leq 1, \\ x - 1, & \text{if } 2 < x \leq 3. \end{cases}$$

Show that f is continuous, one-to-one, and onto, but that f^{-1} is not continuous.

5.4.5 Uniform continuity

Definition 5.4.5. Suppose $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$. We say f is *uniformly continuous* on D if for every $\epsilon > 0$ there exists $\delta > 0$ such that for any $x, y \in D$,

$$|f(x) - f(y)| < \epsilon \text{ whenever } |x - y| < \delta. \tag{5.4.27}$$

Exercise 5.4.22. Suppose $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ is Lipschitz (see Exercise 5.4.10). Show that f is uniformly continuous on D.

Clearly, if f is uniformly continuous on D then f is continuous on D. However, a continuous function need not be uniformly continuous.

Example 5.4.4. Define $f : (0, +\infty)$ by $f(x) = \frac{1}{x}$. Given any $\delta > 0$, choose $n \in \mathbb{Z}^+$ such that $\frac{1}{n(n+1)} < \delta$. Let $x = \frac{1}{n}$ and $y = \frac{1}{n+1}$. Then

$$|x - y| = \frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)} < \delta.$$

However,

$$|f(x) - f(y)| = |n - (n+1)| = 1.$$

Hence, for example, there does not exist a $\delta > 0$ such that

$$|f(x) - f(y)| < \frac{1}{2}$$

whenever $|x - y| < \delta$. Thus f is not uniformly continuous on $(0, +\infty)$, although f is continuous on $(0, +\infty)$.

Example 5.4.5. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = 2x$. Let $\epsilon > 0$ be given. If $\delta = \frac{\epsilon}{2}$, then

$$|f(x) - f(y)| = 2|x - y| < \epsilon$$

whenever $|x - y| < \delta$. Hence f is uniformly continuous on $\mathbb{R}$.

Exercise 5.4.23. Let $f(x) = x^2$. Show that f is not uniformly continuous on $(-\infty, +\infty)$.

Proposition 5.4.13. Suppose $D \subset \mathbb{R}$ is compact and $f : D \to \mathbb{R}$ is continuous. Then f is uniformly continuous on D.

Proof. Let $\epsilon > 0$ be given. For every $x \in D$, choose δ_x such that

$$|f(x) - f(y)| < \frac{\epsilon}{2} \tag{5.4.28}$$

whenever $y \in D$ and $|x - y| < \delta_x$. Let

$$J_x = \left(x - \frac{\delta_x}{2}, x + \frac{\delta_x}{2}\right). \tag{5.4.29}$$

Then $\{J_x : x \in D\}$ is an open cover of D. Since D is compact, there must exist $x_1, x_2, \ldots, x_n$, $n \in Z^+$, such that $J_{x_1}, J_{x_2}, \ldots, J_{x_n}$ is an open cover of D. Let δ be the smallest of

$$\frac{\delta_{x_1}}{2}, \frac{\delta_{x_2}}{2}, \ldots, \frac{\delta_{x_n}}{2}. \tag{5.4.30}$$

Now let $x, y \in D$ with $|x-y| < \delta$. Then for some integer k, $1 \leq k \leq n$, $x \in J_{x_k}$, that is,

$$|x - x_k| < \frac{\delta_{x_k}}{2}. \tag{5.4.31}$$

Moreover,

$$|y - x_k| \leq |y - x| + |x - x_k| < \delta + \frac{\delta_{x_k}}{2} \leq \delta_{x_k}. \tag{5.4.32}$$

Hence

$$|f(x) - f(y)| \leq |f(x) - f(x_k)| + |f(x_k) - f(y)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \tag{5.4.33}$$

Q.E.D.

Exercise 5.4.24. Suppose $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ is uniformly continuous. Show that if $\{x_n\}_{n \in I}$ is a Cauchy sequence in D, then $\{f(x_n)\}_{n \in I}$ is a Cauchy sequence in $f(D)$.

Exercise 5.4.25. Suppose $f : (0,1) \to \mathbb{R}$ is uniformly continuous. Show that $f(0+)$ exists.

Exercise 5.4.26. Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous and $\lim\limits_{x \to -\infty} f(x) = 0$ and $\lim\limits_{x \to +\infty} f(x) = 0$. Show that f is uniformly continuous.

Chapter 6

Derivatives

6.1 Best linear approximations

Definition 6.1.1. We say a function $f : \mathbb{R} \to \mathbb{R}$ is *linear* if for every $x, y \in \mathbb{R}$,

$$f(x + y) = f(x) + f(y) \tag{6.1.1}$$

and for every $\alpha \in \mathbb{R}$ and $x \in \mathbb{R}$,

$$f(\alpha x) = \alpha f(x). \tag{6.1.2}$$

Exercise 6.1.1. Show that if $f : \mathbb{R} \to \mathbb{R}$ is linear, then there exists $m \in \mathbb{R}$ such that $f(x) = mx$ for all $x \in \mathbb{R}$.

Definition 6.1.2. Suppose $D \in \mathbb{R}$, $f : D \to \mathbb{R}$, and a is an interior point of D. We say f is *differentiable* at a if there exists a linear function $df_a : \mathbb{R} \to \mathbb{R}$ such that

$$\lim_{x \to a} \frac{f(x) - f(a) - df_a(x - a)}{x - a} = 0. \tag{6.1.3}$$

We call the function df_a the *best linear approximation* to f at a, or the *differential* of f at a.

Proposition 6.1.1. Suppose $D \subset \mathbb{R}$, $f : D \to \mathbb{R}$, and a is an interior point of D. Then f is differentiable at a if and only if

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} \tag{6.1.4}$$

exists, in which case $df_a(x) = mx$ where

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}. \tag{6.1.5}$$

Proof. Let $m \in \mathbb{R}$ and let $L : \mathbb{R} \to \mathbb{R}$ be the linear function $L(x) = mx$. Then

$$\begin{aligned} \frac{f(x) - f(a) - L(x-a)}{x-a} &= \frac{f(x) - f(a) - m(x-a)}{x-a} \\ &= \frac{f(x) - f(a)}{x-a} - m. \end{aligned} \tag{6.1.6}$$

Hence

$$\lim_{x \to a} \frac{f(x) - f(a) - L(x-a)}{x-a} = 0 \tag{6.1.7}$$

if and only if

$$\lim_{x \to a} \frac{f(x) - f(a)}{x-a} = m. \tag{6.1.8}$$

Q.E.D.

6.2 Derivatives

Definition 6.2.1. Suppose $D \subset \mathbb{R}$, $f : D \to \mathbb{R}$, a is an interior point of D, and f is differentiable at a. We call

$$\lim_{x \to a} \frac{f(x) - f(a)}{x-a} \tag{6.2.1}$$

the *derivative* of f at a, which we denote $f'(a)$.

Note that if f is differentiable at a, then

$$\lim_{x \to a} \frac{f(x) - f(a)}{x-a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}. \tag{6.2.2}$$

Definition 6.2.2. Suppose $D \subset \mathbb{R}$, $f : D \to \mathbb{R}$, and E is the set of interior points of D at which f is differentiable. We call the function $f' : E \to \mathbb{R}$ defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{6.2.3}$$

the *derivative* of f.

Example 6.2.1. Let $n \in \mathbb{Z}^+$ and define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^n$. Then

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{(x+h)^n - x^n}{h} \\ &= \lim_{h \to 0} \frac{x^n + nx^{n-1}h + \sum_{k=2}^{n} \binom{n}{k} x^{n-k} h^k - x^n}{h} \\ &= \lim_{h \to 0} \left(nx^{n-1} + \sum_{k=2}^{n} \binom{n}{k} x^{n-k} h^{k-1} \right) \\ &= nx^{n-1}. \end{aligned}$$

Example 6.2.2. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = |x|$. Then

$$\frac{f(0+h)-f(0)}{h} = \frac{|h|}{h} = \begin{cases} 1, & \text{if } h > 0, \\ -1, & \text{if } h < 0. \end{cases}$$

Hence

$$\lim_{h \to 0^-} \frac{f(0+h)-f(0)}{h} = -1$$

and

$$\lim_{h \to 0^+} \frac{f(0+h)-f(0)}{h} = 1.$$

Thus f is not differentiable at 0.

Exercise 6.2.1. Show that if $c \in \mathbb{R}$ and $f(x) = c$ for all $x \in \mathbb{R}$, then $f'(x) = 0$ for all $x \in \mathbb{R}$.

Exercise 6.2.2. Define $f : [0, +\infty) \to [0, +\infty)$ by $f(x) = \sqrt{x}$. Show that $f' : (0, +\infty) \to (0, +\infty)$ is given by

$$f'(x) = \frac{1}{2\sqrt{x}}.$$

Exercise 6.2.3. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x, & \text{if } x < 0, \\ x^2, & \text{if } x \geq 0. \end{cases}$$

Is f differentiable at 0?

Exercise 6.2.4. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x^2, & \text{if } x < 0, \\ x^3, & \text{if } x \geq 0. \end{cases}$$

Is f differentiable at 0?

Proposition 6.2.1. If f is differentiable at a, then f is continuous at a.

Proof. If f is differentiable at a, then

$$\lim_{x \to a} (f(x) - f(a)) = \lim_{x \to a} \left(\frac{f(x) - f(a)}{x - a} \right) (x - a) = f'(a)(0) = 0. \qquad (6.2.4)$$

Hence $\lim_{x \to a} f(x) = f(a)$, and so f is continuous at a. **Q.E.D.**

6.2.1 The rules

Proposition 6.2.2. Suppose f is differentiable at a and $\alpha \in \mathbb{R}$. Then αf is differentiable at a and $(\alpha f)'(a) = \alpha f'(a)$.

Exercise 6.2.5. Prove the previous proposition.

Proposition 6.2.3. Suppose f and g are both differentiable at a. Then $f + g$ is differentiable at a and $(f + g)'(a) = f'(a) + g'(a)$.

Exercise 6.2.6. Prove the previous proposition.

Proposition 6.2.4 (Product rule)**.** Suppose f and g are both differentiable at a. Then fg is differentiable at a and

$$(fg)'(a) = f(a)g'(a) + g(a)f'(a). \tag{6.2.5}$$

Proof. We have

$$\begin{aligned}(fg)'(a) &= \lim_{h\to 0} \frac{f(a+h)g(a+h) - f(a)g(a)}{h} \\ &= \lim_{h\to 0} \frac{f(a+h)g(a+h) - f(a)g(a+h) + f(a)g(a+h) - f(a)g(a)}{h} \\ &= \lim_{h\to 0} \left(g(a+h)\frac{f(a+h) - f(a)}{h} + f(a)\frac{g(a+h) - g(a)}{h} \right) \\ &= g(a)f'(a) + f(a)g'(a),\end{aligned} \tag{6.2.6}$$

where we know $\lim_{h\to 0} g(a+h) = g(a)$ by the continuity of g at a, which in turn follows from the assumption that g is differentiable at a. **Q.E.D.**

Exercise 6.2.7. Given $n \in \mathbb{Z}^+$ and $f(x) = x^n$, use induction and the product rule to show that $f'(x) = nx^{n-1}$.

Proposition 6.2.5 (Quotient rule)**.** Suppose $D \subset \mathbb{R}$, $f : D \to \mathbb{R}$, $g : D \to \mathbb{R}$, a is in the interior of D, and $g(x) \neq 0$ for all $x \in D$. If f and g are both differentiable at a, then $\frac{f}{g}$ is differentiable at a and

$$\left(\frac{f}{g}\right)'(a) = \frac{g(a)f'(a) - f(a)g'(a)}{(g(a))^2}. \tag{6.2.7}$$

Proof. We have

$$\begin{aligned}
\left(\frac{f}{g}\right)'(a) &= \lim_{h\to 0}\frac{\dfrac{f(a+h)}{g(a+h)}-\dfrac{f(a)}{g(a)}}{h}\\
&= \lim_{h\to 0}\frac{f(a+h)g(a)-f(a)g(a+h)}{hg(a+h)g(a)}\\
&= \lim_{h\to 0}\frac{f(a+h)g(a)-f(a)g(a)+f(a)g(a)-f(a)g(a+h)}{hg(a+h)g(a)}\\
&= \lim_{h\to 0}\frac{g(a)\dfrac{f(a+h)-f(a)}{h}-f(a)\dfrac{g(a+h)-g(a)}{h}}{g(a+h)g(a)}\\
&= \frac{g(a)f'(a)-f(a)g'(a)}{(g(a))^2},
\end{aligned}\tag{6.2.8}$$

where we know $\lim_{h\to 0} g(a+h) = g(a)$ by the continuity of g at a, which in turn follows from the assumption that g is differentiable at a. **Q.E.D.**

Exercise 6.2.8. Show that for any integer $n \neq 0$, if $f(x) = x^n$, then $f'(x) = nx^{n-1}$.

Proposition 6.2.6 (Chain rule)**.** Suppose $D \subset \mathbb{R}$, $E \subset \mathbb{R}$, $g : D \to \mathbb{R}$, $f : E \to \mathbb{R}$, $g(D) \subset E$, g is differentiable at a, and f is differentiable at $g(a)$. Then $f \circ g$ is differentiable at a and

$$(f \circ g)'(a) = f'(g(a))g'(a). \tag{6.2.9}$$

Proof. Since a is an interior point of D and $g(a)$ is an interior point of E, we may choose $\delta > 0$ so that $(a-\delta, a+\delta) \subset D$ and $\epsilon > 0$ so that $(g(a)-\epsilon, g(a)+\epsilon) \subset E$. Define $\varphi : (-\delta, \delta) \to \mathbb{R}$ by

$$\varphi(h) = \begin{cases} \dfrac{g(a+h)-g(a)-g'(a)h}{h}, & \text{if } h \neq 0,\\ 0, & \text{if } h = 0,\end{cases} \tag{6.2.10}$$

and $\psi : (-\epsilon, \epsilon) \to \mathbb{R}$ by

$$\psi(h) = \begin{cases} \dfrac{f(g(a)+h)-f(g(a))-f'(g(a))h}{h}, & \text{if } h \neq 0,\\ 0, & \text{if } h = 0.\end{cases} \tag{6.2.11}$$

The assumption that g is differentiable at a implies that φ is continuous at 0 and the assumption that f is differentiable at $g(a)$ implies that ψ is continuous at 0. Moreover, note that

$$g(a+h) = h\varphi(h) + g'(a)h + g(a) \tag{6.2.12}$$

for $h \in (-\delta, \delta)$ and

$$f(g(a) + h) = h\psi(h) + f'(g(a))h + f(g(a)) \tag{6.2.13}$$

for $h \in (-\epsilon, \epsilon)$. From (6.2.12) we have

$$f(g(a+h)) = f(h\varphi(h) + g'(a)h + g(a)) \tag{6.2.14}$$

for $h \in (-\delta, \delta)$. Now

$$\lim_{h \to 0}(h\varphi(h) + g'(a)h) = 0, \tag{6.2.15}$$

so we may choose $\gamma > 0$ so that $\gamma \leq \delta$ and

$$|h\varphi(h) + g'(a)h| < \epsilon \tag{6.2.16}$$

whenever $h \in (-\gamma, \gamma)$. Thus, using (6.2.13) and (6.2.14),

$$\begin{aligned} f(g(a+h)) = (h\varphi(h) + g'(a)h)&\psi(h\varphi(h) + g'(a)h) \\ &+ f'(g(a))(h\varphi(h) + g'(a)h) + f(g(a)), \end{aligned} \tag{6.2.17}$$

so

$$\begin{aligned} f(g(a+h)) - f(g(a)) &= (h\varphi(h) + g'(a)h)\psi(h\varphi(h) + g'(a)h) \\ &\quad + f'(g(a))(h\varphi(h) + g'(a)h) \\ &= h\varphi(h)\psi(h\varphi(h) + g'(a)h) \\ &\quad + hg'(a)\psi(h\varphi(h) + g'(a)h) \\ &\quad + f'(g(a))\varphi(h)h + f'(g(a))g'(a)h. \end{aligned} \tag{6.2.18}$$

Hence

$$\begin{aligned} \frac{f(g(a+h)) - f(g(a))}{h} &= f'(g(a))g'(a) + \varphi(h)\psi(h\varphi(h) + g'(a)h) \\ &\quad + g'(a)\psi(h\varphi(h) + g'(a)h) + f'(g(a))\varphi(h). \end{aligned} \tag{6.2.19}$$

Now

$$\lim_{h \to 0} \varphi(h) = 0, \tag{6.2.20}$$

$$\lim_{h \to 0}(h\varphi(h) + g'(a)h) = 0, \tag{6.2.21}$$

and, since φ and ψ are continuous at 0,

$$\lim_{h \to 0} \psi(h\varphi(h) + g'(a)h) = 0. \tag{6.2.22}$$

Thus

$$\lim_{h \to 0} \frac{f(g(a+h)) - f(g(a))}{h} = f'(g(a))g'(a). \tag{6.2.23}$$

Q.E.D.

Proposition 6.2.7. Suppose $D \subset \mathbb{R}$, $f : D \to \mathbb{R}$ is one-to-one, a is in the interior of D, $f(a)$ is in the interior of $f(D)$, f^{-1} is continuous at $f(a)$, and f is differentiable at a with $f'(a) \neq 0$. Then f^{-1} is differentiable at $f(a)$ and

$$(f^{-1})'(f(a)) = \frac{1}{f'(a)}. \tag{6.2.24}$$

Proof. Choose $\delta > 0$ so that $(f(a) - \delta, f(a) + \delta) \subset f(D)$. For $h \in (-\delta, \delta)$, let

$$k = f^{-1}(f(a) + h) - a. \tag{6.2.25}$$

Then

$$f^{-1}(f(a) + h) = a + k, \tag{6.2.26}$$

so

$$f(a) + h = f(a + k) \tag{6.2.27}$$

and

$$h = f(a + k) - f(a). \tag{6.2.28}$$

Hence

$$\frac{f^{-1}(f(a) + h) - f^{-1}(f(a))}{h} = \frac{a + k - a}{f(a + k) - f(a)} = \frac{1}{\frac{f(a + k) - f(a)}{k}}. \tag{6.2.29}$$

Now if $h \to 0$, then $k \to 0$ (since f^{-1} is continuous at $f(a)$), and so

$$\lim_{h \to 0} \frac{f^{-1}(f(a) + h) - f^{-1}(f(a))}{h} = \lim_{k \to 0} \frac{1}{\frac{f(a + k) - f(a)}{k}} = \frac{1}{f'(a)}. \tag{6.2.30}$$

Q.E.D.

Example 6.2.3. For $n \in Z^+$, define $f : [0, +\infty) \to \mathbb{R}$ by $f(x) = \sqrt[n]{x}$. Then f is the inverse of $g : [0, +\infty) \to \mathbb{R}$ defined by $g(x) = x^n$. Thus, for any $x \in (0, +\infty)$,

$$f'(x) = \frac{1}{g'(f(x))} = \frac{1}{n(\sqrt[n]{x})^{n-1}} = \frac{1}{n} x^{\frac{1}{n} - 1}.$$

Exercise 6.2.9. Let $n \neq 0$ be a rational number and let $f(x) = x^n$. Show that $f'(x) = nx^{n-1}$.

6.3 Mean Value Theorem

6.3.1 Rolle's theorem

Definition 6.3.1. We say f is *differentiable* on an open interval I if f is differentiable at every point $a \in I$.

Definition 6.3.2. Suppose $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$. We say f has a *local maximum* at a point $a \in D$ if there exists $\delta > 0$ such that $f(a) \geq f(x)$ for all $x \in (a - \delta, a + \delta) \cap D$. We say f has a *local minimum* at a point $a \in D$ if there exists $\delta > 0$ such that $f(a) \leq f(x)$ for all $x \in (a - \delta, a + \delta) \cap D$.

Proposition 6.3.1. Suppose $D \subset \mathbb{R}$, $f : D \to \mathbb{R}$, and a is an interior point of D at which f has either a local maximum or a local minimum. If f is differentiable at a, then $f'(a) = 0$.

Proof. Suppose f has a local maximum at a (a similar argument works if f has a local minimum at a). Choose $\delta > 0$ so that $(a - \delta, a + \delta) \subset D$ and $f(a) \geq f(x)$ for all $x \in (a - \delta, a + \delta)$. Then

$$\frac{f(x) - f(a)}{x - a} \geq 0 \tag{6.3.1}$$

for all $x \in (a - \delta, a)$ and

$$\frac{f(x) - f(a)}{x - a} \leq 0 \tag{6.3.2}$$

for all $x \in (a, a + \delta)$. Hence

$$\lim_{x \to a^-} \frac{f(x) - f(a)}{x - a} \geq 0 \tag{6.3.3}$$

and

$$\lim_{x \to a^+} \frac{f(x) - f(a)}{x - a} \leq 0. \tag{6.3.4}$$

Hence

$$0 \leq \lim_{x \to a^-} \frac{f(x) - f(a)}{x - a} = f'(a) = \lim_{x \to a^+} \frac{f(x) - f(a)}{x - a} \leq 0, \tag{6.3.5}$$

so we must have $f'(a) = 0$. **Q.E.D.**

Theorem 6.3.2 (Rolle's Theorem)**.** Let $a, b \in \mathbb{R}$ and suppose f is continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$, then there exists a point $c \in (a, b)$ at which $f'(c) = 0$.

Proof. By the Extreme Value Theorem, we know f attains a maximum and a minimum value on $[a, b]$. Let m be the minimum value and M the maximum value of f on $[a, b]$. If $m = M = f(a) = f(b)$, then $f(x) = m$ for all $x \in [a, b]$, and so $f'(x) = 0$ for all $x \in (a, b)$. Otherwise, one of m or M occurs at a point c in (a, b). Hence f has either a local maximum or a local minimum at c, and so $f'(c) = 0$. **Q.E.D.**

Exercise 6.3.1. Suppose f is differentiable on (a, b) and $f'(x) \neq 0$ for all $x \in (a, b)$. Show that for any $x, y \in (a, b)$, $f(x) \neq f(y)$.

Exercise 6.3.2. Explain why the equation $x^5 + 10x = 5$ has exactly one solution.

Exercise 6.3.3. Let $f(x)$ be a third degree polynomial. Show that the equation $f(x) = 0$ as at least one, but no more than three, solutions.

6.3.2 Mean Value Theorem

Theorem 6.3.3 (Generalized Mean Value Theorem)**.** Let $a, b \in \mathbb{R}$. If f and g are continuous on $[a, b]$ and differentiable on (a, b), then there exists a point $c \in (a, b)$ at which

$$(f(b) - f(a))g'(c) = (g(b) - g(a))f'(c). \tag{6.3.6}$$

Proof. Let

$$h(t) = (f(b) - f(a))g(t) - (g(b) - g(a))f(t). \tag{6.3.7}$$

Then h is continuous on $[a, b]$ and differentiable on (a, b). Moreover,

$$\begin{aligned} h(a) &= f(b)g(a) - f(a)g(a) - f(a)g(b) + f(a)g(a) \\ &= f(b)g(a) - f(a)g(b) \end{aligned} \tag{6.3.8}$$

and

$$\begin{aligned} h(b) &= f(b)g(b) - f(a)g(b) - f(b)g(b) + f(b)g(a) \\ &= f(b)g(a) - f(a)g(b). \end{aligned} \tag{6.3.9}$$

Hence, by Rolle's theorem, there exists a point $c \in (a, b)$ at which $h'(c) = 0$. But then

$$0 = h'(c) = (f(b) - f(a))g'(c) - (g(b) - g(a))f'(c), \tag{6.3.10}$$

which implies that

$$(f(b) - f(a))g'(c) = (g(b) - g(a))f'(c). \tag{6.3.11}$$

Q.E.D.

Theorem 6.3.4 (Mean Value Theorem)**.** Let $a, b \in \mathbb{R}$. If f is continuous on $[a, b]$ and differentiable on (a, b), then there exists a point $c \in (a, b)$ at which

$$f(b) - f(a) = (b - a)f'(c). \tag{6.3.12}$$

Proof. Apply the previous result with $g(x) = x$. **Q.E.D.**

Exercise 6.3.4. Prove the Mean Value Theorem using Rolle's theorem and the function

$$k(t) = f(t) - \left(\left(\frac{f(b) - f(a)}{b - a} \right) (t - a) + f(a) \right).$$

Give a geometric interpretation for k and compare it with the function h used in the proof of the generalized mean value theorem.

Exercise 6.3.5. Let $a, b \in \mathbb{R}$. Suppose f is continuous on $[a, b]$, differentiable on (a, b), and $|f'(x)| \leq M$ for all $x \in (a, b)$. Show that

$$|f(b) - f(a)| \leq M|b - a|.$$

Exercise 6.3.6. Show that for all $x > 0$,

$$\sqrt{1 + x} < 1 + \frac{x}{2}.$$

Exercise 6.3.7. Suppose I is an open interval, $f : I \to \mathbb{R}$, and $f'(x) = 0$ for all $x \in I$. Show that there exists $\alpha \in \mathbb{R}$ such that $f(x) = \alpha$ for all $x \in I$.

Exercise 6.3.8. Suppose I is an open interval, $f : I \to \mathbb{R}$, $g : I \to \mathbb{R}$, and $f'(x) = g'(x)$ for all $x \in I$. Show that there exists $\alpha \in \mathbb{R}$ such that

$$g(x) = f(x) + \alpha$$

for all $x \in I$.

Exercise 6.3.9. Let $D = \mathbb{R} \setminus \{0\}$. Define $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ by $f(x) = x^2$ and

$$g(x) = \begin{cases} x^2, & \text{if } x < 0, \\ x^2 + 1, & \text{if } x > 0. \end{cases}$$

Show that $f'(x) = g'(x)$ for all $x \in D$, but there does not exist $\alpha \in \mathbb{R}$ such that $g(x) = f(x) + \alpha$ for all $x \in D$. Why does this not contradict the conclusion of the previous exercise?

Proposition 6.3.5. If f is differentiable on (a, b) and $f'(x) > 0$ for all $x \in (a, b)$, then f is increasing on (a, b).

Proof. Let $x, y \in (a, b)$ with $x < y$. By the Mean Value Theorem, there exists a point $c \in (x, y)$ such that

$$f(y) - f(x) = (y - x)f'(c). \tag{6.3.13}$$

Since $y - x > 0$ and $f'(c) > 0$, we have $f(y) > f(x)$, and so f is increasing on (a, b). **Q.E.D.**

Proposition 6.3.6. If f is differentiable on (a, b) and $f'(x) < 0$ for all $x \in (a, b)$, then f is decreasing on (a, b).

Exercise 6.3.10. State and prove similar conditions for nonincreasing and nondecreasing functions.

6.4 Discontinuities of derivatives

Theorem 6.4.1 (Intermediate Value Theorem for Derivatives)**.** Suppose f is differentiable on an open interval I, $a, b \in I$, and $a < b$. If $\lambda \in \mathbb{R}$ and either $f'(a) < \lambda < f'(b)$ or $f'(a) > \lambda > f'(b)$, then there exists $c \in (a, b)$ such that $f'(c) = \lambda$.

Proof. Suppose $f'(a) < \lambda < f'(b)$ and define $g : I \to \mathbb{R}$ by $g(x) = f(x) - \lambda x$. Then g is differentiable on I, and so continuous on $[a, b]$. Let c be a point in $[a, b]$ at which g attains its minimum value. Now

$$g'(a) = f'(a) - \lambda < 0, \tag{6.4.1}$$

so there exists $a < t < b$ such that

$$g(t) - g(a) < 0. \tag{6.4.2}$$

Thus $c \neq a$. Similarly,

$$g'(b) = f'(b) - \lambda > 0, \tag{6.4.3}$$

so there exists $a < s < b$ such that

$$g(s) - g(b) < 0. \tag{6.4.4}$$

Thus $c \neq b$. Hence $c \in (a, b)$, and so $g'(c) = 0$. Thus $0 = f'(c) - \lambda$, and so $f'(c) = \lambda$. **Q.E.D.**

Exercise 6.4.1. Define $g : (-1, 1) \to \mathbb{R}$ by

$$g(x) = \begin{cases} -1, & \text{if } -1 < x < 0, \\ 1, & \text{if } 0 \leq x < 1. \end{cases}$$

Does there exist a function $f : (-1, 1) \to \mathbb{R}$ such that $f'(x) = g(x)$ for all $x \in (-1, 1)$?

Exercise 6.4.2. Suppose f is differentiable on an open interval I. Show that f' cannot have any simple discontinuities in I.

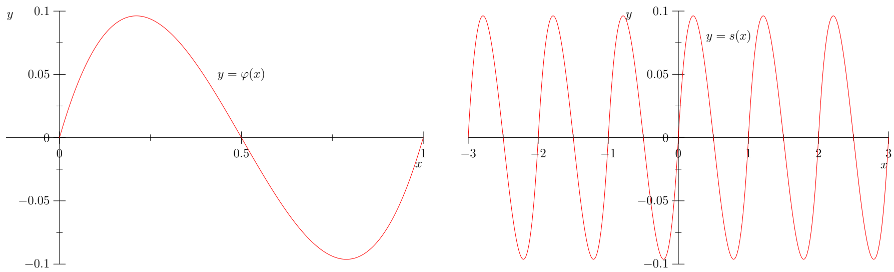

Figure 6.4.1: Graphs of $y = \varphi(x)$ and $y = s(x)$

Example 6.4.1. Define $\varphi : [0, 1] \to \mathbb{R}$ by $\varphi(x) = x(2x - 1)(x - 1)$. Define $\rho : \mathbb{R} \to \mathbb{R}$ by $\rho(x) = 6x^2 - 6x + 1$. Then

$$\varphi(x) = 2x^3 - 3x^2 + x,$$

so $\varphi'(x) = \rho(x)$ for all $x \in (0, 1)$. Next define $s : \mathbb{R} \to \mathbb{R}$ by $s(x) = \varphi(x - \lfloor x \rfloor)$. See Figure 6.4.1 for the graphs of φ and s. Then for any $n \in \mathbb{Z}$ and $n < x < n+1$,

$$s'(x) = \rho(x - n) = \rho(x - \lfloor x \rfloor).$$

Moreover, if x is an integer,

$$\begin{aligned}
\lim_{h\to 0^+} \frac{s(x+h) - s(x)}{h} &= \lim_{h\to 0^+} \frac{\varphi(h)}{h} \\
&= \lim_{h\to 0^+} \frac{h(2h-1)(h-1)}{h} \\
&= \lim_{h\to 0^+} (2h-1)(h-1) \\
&= 1
\end{aligned}$$

and

$$\begin{aligned}
\lim_{h\to 0^-} \frac{s(x+h) - s(x)}{h} &= \lim_{h\to 0^-} \frac{\varphi(h+1)}{h} \\
&= \lim_{h\to 0^-} \frac{(h+1)(2h+1)h}{h} \\
&= \lim_{h\to 0^-} (h+1)(2h+1) \\
&= 1.
\end{aligned}$$

Thus $s'(x) = 1 = \rho(x - \lfloor x \rfloor)$ when x is an integer, and so $s'(x) = \rho(x - \lfloor x \rfloor)$ for all $x \in \mathbb{R}$.

Now $\rho(x) = 0$ if and only if $x = \frac{3-\sqrt{3}}{6}$ or $x = \frac{3+\sqrt{3}}{6}$. Since $\varphi(0) = 0$, $\varphi\left(\frac{3-\sqrt{3}}{6}\right) = \frac{1}{6\sqrt{3}}$, $\varphi\left(\frac{3+\sqrt{3}}{6}\right) = -\frac{1}{6\sqrt{3}}$, and $\varphi(1) = 0$, we see that φ attains a

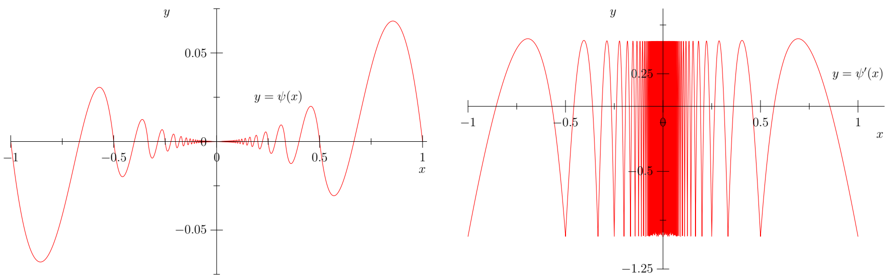

Figure 6.4.2: Graphs of $y = \psi(x)$ and $y = \psi'(x)$

maximum value of $\frac{1}{6\sqrt{3}}$ and a minimum value of $-\frac{1}{6\sqrt{3}}$. Hence for any $n \in \mathbb{Z}$,

$$s((n, n+1)) = \left[-\frac{1}{6\sqrt{3}}, \frac{1}{6\sqrt{3}}\right].$$

Also, $\rho'(x) = 12x - 6$, so $\rho'(x) = 0$ if and only if $x = \frac{1}{2}$. Since $\rho(0) = 1$, $\rho\left(\frac{1}{2}\right) = -\frac{1}{2}$, and $\rho(1) = 1$, we see that ρ attains a maximum value of 1 and a minimum value of $-\frac{1}{2}$ on the interval $[0, 1]$. Hence for any $n \in \mathbb{Z}$,

$$s'((n, n+1)) = \left[-\frac{1}{2}, 1\right].$$

It follows from the preceding, in the same manner as the result in Example 5.1.7, that neither the function $\sigma(x) = s\left(\frac{1}{x}\right)$ nor the function $g(x) = s'\left(\frac{1}{x}\right)$ has a limit as x approaches 0.

Finally, define $\psi : \mathbb{R} \to \mathbb{R}$ by

$$\psi(x) = \begin{cases} x^2 s\left(\dfrac{1}{x}\right), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

For $x \neq 0$, we have

$$\psi'(x) = x^2 s'\left(\frac{1}{x}\right)\left(-\frac{1}{x^2}\right) + 2xs\left(\frac{1}{x}\right) = -s'\left(\frac{1}{x}\right) + 2xs\left(\frac{1}{x}\right).$$

At 0, we have

$$\begin{aligned} \psi'(0) &= \lim_{h\to 0} \frac{\psi(0+h) - \psi(0)}{h} \\ &= \lim_{h\to 0} \frac{h^2 s\left(\frac{1}{h}\right)}{h} \\ &= \lim_{h\to 0} hs\left(\frac{1}{h}\right) \\ &= 0, \end{aligned}$$

where the final limit follows from the squeeze theorem and the fact that s is bounded. Hence we see that ψ is continuous on $\mathbb{R}$ and differentiable on $\mathbb{R}$, but ψ' is not continuous since $\psi'(x)$ does not have a limit as x approaches 0. See Figure 6.4.2 for the graphs of ψ and ψ'.

Exercise 6.4.3. Let s be as above and define $g : \mathbb{R} \to \mathbb{R}$ by

$$g(x) = \begin{cases} x^4 s\left(\frac{1}{x}\right), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

Show that g is differentiable on $\mathbb{R}$ and that g' is continuous on $\mathbb{R}$.

6.5 l'Hôpital's rule

The following result is one case of *l'Hôpital's rule*.

Theorem 6.5.1. Suppose $a, b \in \mathbb{R}$, f and g are differentiable on (a, b), $g'(x) \neq 0$ for all $x \in (a, b)$, and

$$\lim_{x \to a^+} \frac{f'(x)}{g'(x)} = \lambda. \tag{6.5.1}$$

If $\lim_{x \to a^+} f(x) = 0$ and $\lim_{x \to a^+} g(x) = 0$, then

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = \lambda. \tag{6.5.2}$$

Proof. Given $\epsilon > 0$, there exists $\delta > 0$ such that

$$\lambda - \frac{\epsilon}{2} < \frac{f'(x)}{g'(x)} < \lambda + \frac{\epsilon}{2} \tag{6.5.3}$$

whenever $x \in (a, a+\delta)$. Now, by the Generalized Mean Value Theorem, for any x and y with $a < x < y < a + \delta$, there exists a point $c \in (x, y)$ such that

$$\frac{f(y) - f(x)}{g(y) - g(x)} = \frac{f'(c)}{g'(c)}. \tag{6.5.4}$$

Hence

$$\lambda - \frac{\epsilon}{2} < \frac{f(y) - f(x)}{g(y) - g(x)} < \lambda + \frac{\epsilon}{2}. \tag{6.5.5}$$

Now

$$\lim_{x \to a^+} \frac{f(y) - f(x)}{g(y) - g(x)} = \frac{f(y)}{g(y)}, \tag{6.5.6}$$

and so we have

$$\lambda - \epsilon < \lambda - \frac{\epsilon}{2} \leq \frac{f(y)}{g(y)} \leq \lambda + \frac{\epsilon}{2} < \lambda + \epsilon \tag{6.5.7}$$

for any $y \in (a, a+\delta)$. Hence

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = \lambda. \tag{6.5.8}$$

Q.E.D.

Exercise 6.5.1. Use l'Hôpital's rule to compute

$$\lim_{x \to 0^+} \frac{\sqrt{1+x} - 1}{x}.$$

Exercise 6.5.2. Suppose $a, b \in \mathbb{R}$, f and g are differentiable on (a, b), $g'(x) \neq 0$ for all $x \in (a, b)$, and

$$\lim_{x \to b^-} \frac{f'(x)}{g'(x)} = \lambda.$$

Show that if $\lim_{x \to b^-} f(x) = 0$ and $\lim_{x \to b^-} g(x) = 0$, then

$$\lim_{x \to b^-} \frac{f(x)}{g(x)} = \lambda.$$

6.6 Taylor's Theorem

6.6.1 Derivatives of higher order

Definition 6.6.1. Suppose f is differentiable on an open interval I and f' is differentiable at $a \in I$. We call the derivative of f' at a the *second derivative* of f at a, which we denote $f''(a)$.

By continued differentiation, we may define the *higher order derivatives* f''', f'''', and so on. In general, for any integer n, $n \geq 0$, we let $f^{(n)}$ denote the nth derivative of f, where $f^{(0)}$ denotes f.

Exercise 6.6.1. Suppose $D \subset \mathbb{R}$, a is an interior point of D, $f : D \to \mathbb{R}$, and $f''(a)$ exists. Show that

$$\lim_{h \to 0} \frac{f(a+h) + f(a-h) - 2f(a)}{h^2} = f''(a).$$

Find an example to illustrate that this limit may exist even if $f''(a)$ does not exist.

For any open interval (a, b), where a and b are extended real numbers, we let $C^{(n)}(a, b)$, where $n \in \mathbb{Z}^+$, denote the set of all functions f with the property that each of $f, f^{(1)}, f^{(2)}, \ldots, f^{(n)}$ is defined and continuous on (a, b).

6.6.2 Taylor's Theorem

Theorem 6.6.1 (Taylor's Theorem)**.** Suppose $f \in C^{(n)}(a,b)$ and $f^{(n)}$ is differentiable on (a,b). Let $\alpha, \beta \in (a,b)$ with $\alpha \neq \beta$, and let

$$\begin{aligned} P(x) &= f(\alpha) + f'(\alpha)(x-\alpha) + \frac{f''(\alpha)}{2}(x-\alpha)^2 + \cdots \\ &\quad + \frac{f^{(n)}(\alpha)}{n!}(x-\alpha)^n \\ &= \sum_{k=0}^{n} \frac{f^{(k)}(\alpha)}{k!}(x-\alpha)^k. \end{aligned} \tag{6.6.1}$$

Then there exists a point γ between α and β such that

$$f(\beta) = P(\beta) + \frac{f^{(n+1)}(\gamma)}{(n+1)!}(\beta-\alpha)^{n+1}. \tag{6.6.2}$$

Proof. First note that $P^{(k)}(\alpha) = f^{(k)}(\alpha)$ for $k = 0, 1, \ldots, n$. Let

$$M = \frac{f(\beta) - P(\beta)}{(\beta-\alpha)^{n+1}}. \tag{6.6.3}$$

Then

$$f(\beta) = P(\beta) + M(\beta-\alpha)^{n+1}. \tag{6.6.4}$$

We need to show that

$$M = \frac{f^{(n+1)}(\gamma)}{(n+1)!} \tag{6.6.5}$$

for some γ between α and β. Let

$$g(x) = f(x) - P(x) - M(x-\alpha)^{n+1}. \tag{6.6.6}$$

Then, for $k = 0, 1, \ldots, n$,

$$g^{(k)}(\alpha) = f^{(k)}(\alpha) - P^{(k)}(\alpha) = 0. \tag{6.6.7}$$

Now $g(\beta) = 0$, so, by Rolle's theorem, there exists γ_1 between α and β such that $g'(\gamma_1) = 0$. Using Rolle's theorem again, we see that there exists γ_2 between α and γ_1 such that $g''(\gamma_2) = 0$. Continuing for $n+1$ steps, we find γ_{n+1} between α and γ_n (and hence between α and β) such that $g^{(n+1)}(\gamma_{n+1}) = 0$. Hence

$$0 = g^{(n+1)}(\gamma_{n+1}) = f^{(n+1)}(\gamma_{n+1}) - (n+1)!M. \tag{6.6.8}$$

Letting $\gamma = \gamma_{n+1}$, we have

$$M = \frac{f^{(n+1)}(\gamma)}{(n+1)!}, \tag{6.6.9}$$

as required. **Q.E.D.**

We call the polynomial P in the statement of Taylor's theorem the *Taylor polynomial* of order n for f at α.

Example 6.6.1. Let $f(x) = \sqrt{x}$. Then the 4th order Taylor polynomial for f at 1 is

$$P(x) = 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3 - \frac{5}{128}(x-1)^4.$$

By Taylor's theorem, for any $x > 0$ there exists γ between 1 and x such that

$$\sqrt{x} = P(x) + \frac{105}{(32)(5!)\gamma^{\frac{9}{2}}}(x-1)^5 = P(x) + \frac{7}{256\gamma^{\frac{9}{2}}}(x-1)^5.$$

For example,

$$\sqrt{1.2} = P(1.2) + \frac{7}{256\gamma^{\frac{9}{2}}}(1.2-1)^5 = P(1.2) + \frac{7}{256\gamma^{\frac{9}{2}}}(0.2)^5 = P(1.2) + \frac{7}{800000\gamma^{\frac{9}{2}}},$$

for some γ with $1 < \gamma < 1.2$. Hence $P(1.2)$ underestimates $\sqrt{1.2}$ by a value which is no larger than $\frac{7}{800000}$. Note that

$$P(1.2) = \frac{17527}{16000} = 1.0954375$$

and

$$\frac{7}{800000} = 0.00000875.$$

So $\sqrt{1.2}$ lies between 1.0954375 and 1.09544625.

Exercise 6.6.2. Use the 5th order Taylor polynomial for $f(x) = \sqrt{x}$ at 1 to estimate $\sqrt{1.2}$. Is this an underestimate or an overestimate? Find an upper bound for the largest amount by which the estimate and $\sqrt{1.2}$ differ.

Exercise 6.6.3. Find the 3rd order Taylor polynomial for $f(x) = \sqrt[3]{1+x}$ at 0 and use it to estimate $\sqrt[3]{1.1}$. Is this an underestimate or an overestimate? Find an upper bound for the largest amount by which the estimate and $\sqrt[3]{1.1}$ differ.

Exercise 6.6.4. Suppose $f \in C^{(2)}(a,b)$. Use Taylor's theorem to show that

$$\lim_{h \to 0} \frac{f(c+h) + f(c-h) - 2f(c)}{h^2} = f''(c)$$

for any $c \in (a,b)$.

Exercise 6.6.5. Suppose $f \in C^{(1)}(a,b)$, $c \in (a,b)$, $f'(c) = 0$, and f'' exists on (a,b) and is continuous at c. Show that f has a local maximum at c if $f''(c) < 0$ and a local minimum at c if $f''(c) > 0$.

Chapter 7

Integrals

7.1 Upper and lower integrals

Definition 7.1.1. Given a closed interval $[a, b] \subset \mathbb{R}$, we call any finite subset of $[a, b]$ which includes both a and b a *partition* of $[a, b]$.

For convenience, whenever we consider a partition P of an interval $[a, b]$ we will index the elements in increasing order, starting with 0. That is, if $|P| = n+1$ and $P = \{x_0, x_1, \ldots, x_n\}$, then

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b. \tag{7.1.1}$$

Definition 7.1.2. Suppose $P = \{x_0, x_1, \ldots, x_n\}$ is a partition of $[a, b]$ and $f : [a, b] \to \mathbb{R}$ is bounded. For $i = 1, 2, \ldots, n$, let

$$m_i = \inf\{f(x) : x_{i-1} \leq x \leq x_i\} \tag{7.1.2}$$

and

$$M_i = \sup\{f(x) : x_{i-1} \leq x \leq x_i\}. \tag{7.1.3}$$

We call

$$L(f, P) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}) \tag{7.1.4}$$

the *lower sum* of f determined by P and

$$U(f, P) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}) \tag{7.1.5}$$

the *upper sum* of f determined by P.

Definition 7.1.3. If P_1 and P_2 are both partitions of $[a, b]$ and $P_1 \subset P_2$, then we call P_2 a *refinement* of P_1.

Definition 7.1.4. f P_1 and P_2 are both partitions of $[a, b]$, then we call the partition $P = P_1 \cup P_2$ the *common refinement* of P_1 and P_2.

Lemma 7.1.1. Suppose $f : [a, b] \to \mathbb{R}$ is bounded and $P_1 = \{x_0, x_1, \ldots, x_n\}$ is a partition of $[a, b]$. Moreover, suppose $s \in (a, b)$ with $s \notin P_1$. If $P_2 = P_1 \cup \{s\}$, then $L(f, P_1) \leq L(f, P_2)$ and $U(f, P_2) \leq U(f, P_1)$.

Proof. Suppose $x_{i-1} < s < x_i$ and let

$$w_1 = \inf\{f(x) : x_{i-1} \leq x \leq s\}, \tag{7.1.6}$$
$$W_1 = \sup\{f(x) : x_{i-1} \leq x \leq s\}, \tag{7.1.7}$$
$$w_2 = \inf\{f(x) : s \leq x \leq x_i\}, \tag{7.1.8}$$
$$W_2 = \sup\{f(x) : s \leq x \leq x_i\}, \tag{7.1.9}$$
$$m_i = \inf\{f(x) : x_{i-1} \leq x \leq x_i\}, \tag{7.1.10}$$

and

$$M_i = \sup\{f(x) : x_{i-1} \leq x \leq x_i\}. \tag{7.1.11}$$

Then $w_1 \geq m_i$, $w_2 \geq m_i$, $W_1 \leq M_i$, and $W_2 \leq M_i$. Hence

$$\begin{aligned} L(f, P_2) - L(f, P_1) &= w_1(s - x_{i-1}) + w_2(x_i - s) - m_i(x_i - x_{i-1}) \\ &= w_1(s - x_{i-1}) + w_2(x_i - s) - m_i(s - x_{i-1}) \\ &\quad - m_i(x_i - s) \\ &= (w_1 - m_i)(s - x_{i-1}) + (w_2 - m_i)(x_i - s) \\ &\geq 0 \end{aligned} \tag{7.1.12}$$

and

$$\begin{aligned} U(f, P_1) - U(f, P_2) &= M_i(x_i - x_{i-1}) - W_1(s - x_{i-1}) - W_2(x_i - s) \\ &= M_i(s - x_{i-1}) + M_i(x_i - s) - W_1(s - x_{i-1}) \\ &\quad - W_2(x_i - s) \\ &= (M_i - W_1)(s - x_{i-1}) + (M_i - W_2)(x_i - s) \\ &\geq 0. \end{aligned} \tag{7.1.13}$$

Thus $L(f, P_1) \leq L(f, P_2)$ and $U(f, P_2) \leq U(f, P_1)$. **Q.E.D.**

Proposition 7.1.2. Suppose $f : [a, b] \to \mathbb{R}$ is bounded and P_1 and P_2 are partitions of $[a, b]$. If P_2 is a refinement of P_1, then $L(f, P_1) \leq L(f, P_2)$ and $U(f, P_2) \leq U(f, P_1)$.

Proof. The proposition follows immediately from repeated use of the previous lemma. **Q.E.D.**

Proposition 7.1.3. Suppose $f : [a, b] \to \mathbb{R}$ is bounded and P_1 and P_2 are partitions of $[a, b]$. Then $L(f, P_1) \leq U(f, P_2)$.

Proof. The result follows immediately from the definitions if $P_1 = P_2$. Otherwise, let P be the common refinement of P_1 and P_2. Then

$$L(f, P_1) \leq L(f, P) \leq U(f, P) \leq U(f, P_2). \tag{7.1.14}$$

Q.E.D.

Definition 7.1.5. Suppose $f : [a, b] \to \mathbb{R}$ is bounded. We call

$$\underline{\int_a^b} f = \sup\{L(f, P) : P \text{ is a partition of } [a, b]\} \tag{7.1.15}$$

the *lower integral* of f over $[a, b]$ and

$$\overline{\int_a^b} f = \inf\{U(f, P) : P \text{ is a partition of } [a, b]\} \tag{7.1.16}$$

the *upper integral* of f over $[a, b]$.

Note that both the lower integral and the upper integral are finite real numbers since the lower sums are all bounded above by any upper sum and the upper sums are all bounded below by any lower sum.

Proposition 7.1.4. Suppose $f : [a, b] \to \mathbb{R}$ is bounded. Then

$$\underline{\int_a^b} f \leq \overline{\int_a^b} f. \tag{7.1.17}$$

Proof. Let P be a partition of $[a, b]$. Then for any partition Q of $[a, b]$, we have $L(f, Q) \leq U(f, P)$. Hence $U(f, P)$ is an upper bound for any lower sum, and so

$$\underline{\int_a^b} f \leq U(f, P). \tag{7.1.18}$$

But this shows that the lower integral is a lower bound for any upper sum. Hence

$$\underline{\int_a^b} f \leq \overline{\int_a^b} f. \tag{7.1.19}$$

Q.E.D.

7.2 Integrals

Definition 7.2.1. Suppose $f : [a,b] \to \mathbb{R}$ is bounded. We say f is *integrable* on $[a,b]$ if

$$\underline{\int_a^b} f = \overline{\int_a^b} f. \tag{7.2.1}$$

If f is integrable, we call the common value of the upper and lower integrals the *integral* of f over $[a,b]$, denoted

$$\int_a^b f. \tag{7.2.2}$$

That is, if f is integrable on $[a,b]$,

$$\int_a^b f = \underline{\int_a^b} f = \overline{\int_a^b} f. \tag{7.2.3}$$

Example 7.2.1. Define $f : [0,1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q}, \\ 0, & \text{if } x \notin \mathbb{Q}. \end{cases}$$

For any partition $P = \{x_0, x_1, \ldots, x_n\}$, we have

$$L(f,P) = \sum_{i=1}^{n} 0(x_i - x_{i-1}) = 0$$

and

$$U(f,P) = \sum_{i=1}^{n} (x_i - x_{i-1}) = x_n - x_0 = 1.$$

Thus

$$\underline{\int_0^1} f = 0$$

and

$$\overline{\int_0^1} f = 1.$$

Hence f is not integrable on $[0,1]$.

Example 7.2.2. Let $\alpha \in \mathbb{R}$ and define $f : [a,b] \to \mathbb{R}$ by $f(x) = \alpha$ for all $x \in [a,b]$. For any partition $P = \{x_0, x_1, \ldots, x_n\}$, we have

$$L(f,P) = \sum_{i=1}^{n} \alpha(x_i - x_{i-1}) = \alpha(x_n - x_0) = \alpha(b-a)$$

and

$$U(f,P) = \sum_{i=1}^{n} \alpha(x_i - x_{i-1}) = \alpha(x_n - x_0) = \alpha(b-a).$$

Thus

$$\underline{\int_a^b} f = \alpha(b-a)$$

and

$$\overline{\int_a^b} f = \alpha(b-a).$$

Hence f is integrable on $[a,b]$ and

$$\int_a^b f = \alpha(b-a).$$

Theorem 7.2.1. Suppose $f : [a,b] \to \mathbb{R}$ is bounded. Then f is integrable on $[a,b]$ if and only if for every $\epsilon > 0$ there exists a partition P of $[a,b]$ such that

$$U(f,P) - L(f,P) < \epsilon. \tag{7.2.4}$$

Proof. If f is integrable on $[a,b]$ and $\epsilon > 0$, then we may choose partitions P_1 and P_2 such that

$$\int_a^b f - L(f,P_1) < \frac{\epsilon}{2} \tag{7.2.5}$$

and

$$U(f,P_2) - \int_a^b f < \frac{\epsilon}{2}. \tag{7.2.6}$$

Let P be the common refinement of P_1 and P_2. Then

$$\begin{aligned} U(f,P) - L(f,P) &\le U(f,P_2) - L(f,P_1) \\ &= \left(U(f,P_2) - \int_a^b f\right) + \left(\int_a^b f - L(f,P_1)\right) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned} \tag{7.2.7}$$

Now suppose for every $\epsilon > 0$ there exists a partition P of $[a,b]$ such that

$$U(f,P) - L(f,P) < \epsilon. \tag{7.2.8}$$

Suppose

$$\underline{\int_a^b} f < \overline{\int_a^b} f. \tag{7.2.9}$$

If

$$\epsilon = \overline{\int_a^b} f - \underline{\int_a^b} f, \tag{7.2.10}$$

then for any partition P of $[a,b]$ we have

$$U(f,P) - L(f,P) \ge \overline{\int_a^b} f - \underline{\int_a^b} f = \epsilon. \tag{7.2.11}$$

Since this contradicts our assumption, we must have

$$\underline{\int_a^b} f = \overline{\int_a^b} f. \tag{7.2.12}$$

That is, f is integrable on $[a, b]$. **Q.E.D.**

Example 7.2.3. Suppose $f : [0, 1] \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} 0, & \text{if } x \neq \frac{1}{2}, \\ 1, & \text{if } x = \frac{1}{2}. \end{cases}$$

If P is a partition of $[0, 1]$, then clearly $L(f, P) = 0$. Given $\epsilon > 0$, let

$$P = \{0, \frac{1}{2} - \frac{\epsilon}{4}, \frac{1}{2} + \frac{\epsilon}{4}, 1\}.$$

Then

$$U(f, P) = \left(\frac{1}{2} + \frac{\epsilon}{4}\right) - \left(\frac{1}{2} - \frac{\epsilon}{4}\right) = \frac{\epsilon}{2} < \epsilon.$$

Hence $U(f, P) - L(f, P) < \epsilon$, so f is integrable on $[0, 1]$. Moreover,

$$\int_0^1 f = 0.$$

Exercise 7.2.1. For $n \in \mathbb{Z}^+$, let $a_1, a_2, \ldots, a_n$ be points in $(0, 1)$. Define $f : [0, 1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1, & \text{if } x = a_i \text{ for some } i, \\ 0, & \text{otherwise.} \end{cases}$$

Show that f is integrable on $[0, 1]$ and

$$\int_0^1 f = 0.$$

Exercise 7.2.2. Define $f : [0, 1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} \frac{1}{q}, & \text{if } x \text{ is rational and } x = \frac{p}{q}, \\ 0, & \text{if } x \text{ is irrational,} \end{cases}$$

where p and q are taken to be relatively prime integers with $q > 0$, and we take $q = 1$ when $x = 0$. Show that f is integrable on $[0, 1]$ and

$$\int_0^1 f = 0.$$

Exercise 7.2.3. Let $f : [0,1] \to \mathbb{R}$ be defined by $f(x) = x$ and, for $n \in \mathbb{Z}^+$, let $P = \{x_0, x_1, \ldots, x_n\}$ be the partition of $[0,1]$ with

$$x_i = \frac{i}{n}, i = 0, 1, \ldots, n.$$

Show that

$$U(f,P) - L(f,P) = \frac{1}{n},$$

and hence conclude that f is integrable on $[0,1]$. Show that

$$\int_0^1 f = \frac{1}{2}.$$

Exercise 7.2.4. Define $f : [1,2] \to \mathbb{R}$ by

$$f(x) = \begin{cases} x, & \text{if } x \in \mathbb{Q}, \\ 0, & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Show that f is not integrable on $[1,2]$.

Exercise 7.2.5. Suppose f is integrable on $[a,b]$, and, for some real number m and M, $m \leq f(x) \leq M$ for all $x \in [a,b]$. Show that

$$m(b-a) \leq \int_a^b f \leq M(b-a).$$

7.2.1 Notation and terminology

The definition of the integral described in this section is due to Darboux. One may show it to be equivalent to the integral defined by Riemann. Hence functions that are integrable in the sense of this discussion are referred to as *Riemann integrable* functions and we call the integral the *Riemann integral*. This is in distinction to the *Lebesgue integral*, part of a more general theory of integration.

We sometimes refer to this integral as the *definite integral*, as opposed to an *indefinite integral*, the latter being a name given to an *antiderivative* (a function whose derivative is equal to a given function).

If f is integrable on $[a,b]$, then we will also denote

$$\int_a^b f \tag{7.2.13}$$

by

$$\int_a^b f(x)dx. \tag{7.2.14}$$

The variable x in the latter is a "dummy" variable; we may just as well write

$$\int_a^b f(t)dt \tag{7.2.15}$$

or

$$\int_a^b f(s)ds. \tag{7.2.16}$$

For example, if $f : [0,1] \to \mathbb{R}$ is defined by $f(x) = x^2$, then

$$\int_0^1 f = \int_0^1 x^2 dx = \int_0^1 t^2 dt. \tag{7.2.17}$$

7.3 Integrability conditions

Proposition 7.3.1. If $f : [a,b] \to \mathbb{R}$ is monotonic, then f is integrable on $[a,b]$.

Proof. Suppose f is nondecreasing. Given $\epsilon > 0$, let $n \in Z^+$ be large enough that

$$\frac{(f(b) - f(a))(b-a)}{n} < \epsilon. \tag{7.3.1}$$

For $i = 0, 1, \ldots, n$, let

$$x_i = a + \frac{(b-a)i}{n}. \tag{7.3.2}$$

Let $P = \{x_0, x_1, \ldots, x_n\}$. Then

$$\begin{aligned} U(f,P) - L(f,P) &= \sum_{i=1}^{n} f(x_i)(x_i - x_{i-1}) - \sum_{i=1}^{n} f(x_{i-1})(x_i - x_{i-1}) \\ &= \sum_{i=1}^{n} (f(x_i) - f(x_{i-1}))\frac{b-a}{n} \\ &= \frac{b-a}{n}((f(x_1) - f(x_0)) + (f(x_2) - f(x_1)) + \cdots \\ &\qquad + (f(x_{n-1}) - f(x_{n-2})) + (f(x_n) - f(x_{n-1}))) \\ &= \frac{b-a}{n}(f(b) - f(a)) \\ &< \epsilon. \end{aligned} \tag{7.3.3}$$

Hence f is integrable on $[a,b]$. **Q.E.D.**

Example 7.3.1. Let $\varphi : \mathbb{Q} \cap [0,1] \to \mathbb{Z}^+$ be a one-to-one correspondence. Define $f : [0,1] \to \mathbb{R}$ by

$$f(x) = \sum_{\substack{q \in \mathbb{Q} \cap [0,1] \\ q \leq x}} \frac{1}{2^{\varphi(q)}}.$$

Then f is increasing on $[0,1]$, and hence integrable on $[0,1]$.

Proposition 7.3.2. If $f : [a, b] \to \mathbb{R}$ is continuous, then f is integrable on $[a, b]$.

Proof. Given $\epsilon > 0$, let

$$\gamma = \frac{\epsilon}{b-a}. \tag{7.3.4}$$

Since f is uniformly continuous on $[a, b]$, we may choose $\delta > 0$ such that

$$|f(x) - f(y)| < \gamma \tag{7.3.5}$$

whenever $|x - y| < \delta$. Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition with

$$\sup\{|x_i - x_{i-1}| : i = 1, 2, \ldots, n\} < \delta. \tag{7.3.6}$$

If, for $i = 1, 2, \ldots, n$,

$$m_i = \inf\{f(x) : x_{i-1} \leq x \leq x_i\} \tag{7.3.7}$$

and

$$M_i = \sup\{f(x) : x_{i-1} \leq x \leq x_i\}, \tag{7.3.8}$$

then $M_i - m_i < \gamma$. Hence

$$\begin{aligned} U(f,P) - L(f,P) &= \sum_{i=1}^{n} M_i(x_i - x_{i-1}) - \sum_{i=1}^{n} m_i(x_i - x_{i-1}) \\ &= \sum_{i=1}^{n} (M_i - m_i)(x_i - x_{i-1}) \\ &< \gamma \sum_{i=1}^{n} (x_i - x_{i-1}) \\ &= \gamma(b-a) \\ &= \epsilon. \end{aligned} \tag{7.3.9}$$

Thus f is integrable on $[a, b]$. **Q.E.D.**

Exercise 7.3.1. Suppose $f : [a, b] \to \mathbb{R}$ is bounded and let $c \in [a, b]$. Show that if f is continuous on $[a, b] \setminus \{c\}$, then f is integrable on $[a, b]$.

Exercise 7.3.2. Suppose f is continuous on $[a, b]$ with $f(x) \geq 0$ for all $x \in [a, b]$. Show that if

$$\int_a^b f = 0,$$

then $f(x) = 0$ for all $x \in [a, b]$.

Exercise 7.3.3. Suppose f is continuous on $[a, b]$. For $i = 0, 1, \ldots, n$, $n \in \mathbb{Z}^+$, let

$$x_i = a + \frac{(b-a)i}{n}$$

and, for $i = 1, 2, \ldots, n$, let $c_i \in [x_{i-1}, x_i]$. Show that

$$\int_a^b f = \lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{n} f(c_i).$$

In the notation of Exercise 7.3.3, we call the approximation

$$\int_a^b f \approx \frac{b-a}{n} \sum_{i=1}^{n} f(c_i) \tag{7.3.10}$$

a *right-hand rule approximation* if $c_i = x_i$, a *left-hand rule approximation* if $c_i = x_{i-1}$, and a *midpoint rule approximation* if

$$c_i = \frac{x_{i-1} + x_i}{2}. \tag{7.3.11}$$

These are basic ingredients in creating numerical approximations to integrals.

7.4 Properties of integrals

Proposition 7.4.1. If $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$, then

$$\sup\{f(x) + g(x) : x \in D\} \le \sup\{f(x) : x \in D\} + \sup\{g(x) : x \in D\} \tag{7.4.1}$$

and

$$\inf\{f(x) + g(x) : x \in D\} \ge \inf\{f(x) : x \in D\} + \inf\{g(x) : x \in D\} \tag{7.4.2}$$

Exercise 7.4.1. Prove the previous proposition.

Exercise 7.4.2. Find examples for which the inequalities in the previous proposition are strict.

Proposition 7.4.2. Suppose f and g are both integrable on $[a, b]$. Then $f + g$ is integrable on $[a, b]$ and

$$\int_a^b (f + g) = \int_a^b f + \int_a^b g. \tag{7.4.3}$$

Proof. Given $\epsilon > 0$, let P_1 and P_2 be partitions of $[a, b]$ with

$$U(f, P_1) - L(f, P_1) < \frac{\epsilon}{2} \tag{7.4.4}$$

and

$$U(g, P_2) - L(g, P_2) < \frac{\epsilon}{2}. \tag{7.4.5}$$

Let $P = P_1 \cup P_2$. By the previous proposition,

$$U(f+g,P) \le U(f,P) + U(g,P) \tag{7.4.6}$$

and

$$L(f+g,P) \ge L(f,P) + L(g,P). \tag{7.4.7}$$

Hence

$$\begin{aligned} U(f+g,P) - L(f+g,P) &\le (U(f,P)+U(g,P)) - (L(f,P)+L(g,P)) \\ &= (U(f,P)-L(f,P)) + (U(g,P)-L(g,P)) \\ &\le (U(f,P_1)-L(f,P_1)) + (U(g,P_2)-L(g,_2P)) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned} \tag{7.4.8}$$

Hence $f+g$ is integrable on $[a,b]$.

Moreover,

$$\begin{aligned} \int_a^b (f+g) &\le U(f+g,P) \\ &\le U(f,P) + U(g,P) \\ &\le \left(\int_a^b f + \frac{\epsilon}{2}\right) + \left(\int_a^b g + \frac{\epsilon}{2}\right) \\ &= \int_a^b f + \int_a^b g + \epsilon \end{aligned} \tag{7.4.9}$$

and

$$\begin{aligned} \int_a^b (f+g) &\ge L(f+g,P) \\ &\ge L(f,P) + L(g,P) \\ &\ge \left(\int_a^b f - \frac{\epsilon}{2}\right) + \left(\int_a^b g - \frac{\epsilon}{2}\right) \\ &= \int_a^b f + \int_a^b g - \epsilon. \end{aligned} \tag{7.4.10}$$

Since $\epsilon > 0$ was arbitrary, it follows that

$$\int_a^b (f+g) = \int_a^b f + \int_a^b g. \tag{7.4.11}$$

Q.E.D.

Exercise 7.4.3. Suppose $f : [a,b] \to \mathbb{R}$ and $g : [a,b] \to \mathbb{R}$ are both bounded. Show that

$$\overline{\int_a^b}(f+g) \leq \overline{\int_a^b} f + \overline{\int_a^b} g.$$

Find an example for which the inequality is strict.

Exercise 7.4.4. Find an example to show that $f+g$ may be integrable on $[a,b]$ even though neither f nor g is integrable on $[a,b]$.

Proposition 7.4.3. If f is integrable on $[a,b]$ and $\alpha \in \mathbb{R}$, then αf is integrable on $[a,b]$ and

$$\int_a^b \alpha f = \alpha \int_a^b f. \tag{7.4.12}$$

Exercise 7.4.5. Prove the previous proposition.

Proposition 7.4.4. Suppose $f : [a,b] \to \mathbb{R}$ is bounded and $c \in (a,b)$. Then f is integrable on $[a,b]$ if and only if f is integrable on both $[a,c]$ and $[c,b]$.

Proof. Suppose f is integrable on $[a,b]$. Given $\epsilon > 0$, let Q be a partition of $[a,b]$ such that

$$U(f,Q) - L(f,Q) < \epsilon. \tag{7.4.13}$$

Let $P = Q \cup \{c\}$, $P_1 = P \cap [a,c]$, and $P_2 = P \cap [c,b]$. Then

$$\begin{aligned}(U(f,P_1) - L(f,P_1)) + (U(f,P_2) - L(f,P_2)) &= (U(f,P_1) + U(f,P_2)) \\ &\quad - (L(f,P_1) + L(f,P_2)) \\ &= U(f,P) - L(f,P) \\ &\leq U(f,Q) - L(f,Q) \\ &< \epsilon.\end{aligned} \tag{7.4.14}$$

Thus we must have both

$$U(f,P_1) - L(f,P_1) < \epsilon \tag{7.4.15}$$

and

$$U(f,P_2) - L(f,P_2) < \epsilon. \tag{7.4.16}$$

Hence f is integrable on both $[a,c]$ and $[c,b]$.

Now suppose f is integrable on both $[a,c]$ and $[c,b]$. Given $\epsilon > 0$, let P_1 and P_2 be partitions of $[a,c]$ and $[c,b]$, respectively, such that

$$U(f,P_1) - L(f,P_1) < \frac{\epsilon}{2} \tag{7.4.17}$$

and

$$U(f, P_2) - L(f, P_2) < \frac{\epsilon}{2}. \tag{7.4.18}$$

Let $P = P_1 \cup P_2$. Then P is a partition of $[a, b]$ and

$$\begin{aligned} U(f, P) - L(f, P) &= (U(f, P_1) + U(f, P_2)) - (L(f, P_1) + L(f, P_2)) \\ &= (U(f, P_1) - L(f, P_1)) + (U(f, P_2) - L(f, P_2)) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned} \tag{7.4.19}$$

Thus f is integrable on $[a, b]$. **Q.E.D.**

Proposition 7.4.5. Suppose f is integrable on $[a, b]$ and $c \in (a, b)$. Then

$$\int_a^b f = \int_a^c f + \int_c^b f. \tag{7.4.20}$$

Proof. If P and Q are partitions of $[a, c]$ and $[c, b]$, respectively, then

$$U(f, P) + U(f, Q) = U(f, P \cup Q) \geq \int_a^b f. \tag{7.4.21}$$

Thus

$$U(f, P) \geq \int_a^b f - U(f, Q), \tag{7.4.22}$$

so

$$\int_a^c f = \overline{\int_a^c} f \geq \int_a^b f - U(f, Q). \tag{7.4.23}$$

Hence

$$U(f, Q) \geq \int_a^b f - \int_a^c f, \tag{7.4.24}$$

so

$$\int_c^b f = \overline{\int_c^b} f \geq \int_a^b f - \int_a^c f. \tag{7.4.25}$$

Thus

$$\int_a^c f + \int_c^b f \geq \int_a^b f. \tag{7.4.26}$$

Similarly, if P and Q are partitions of $[a, c]$ and $[c, b]$, respectively, then

$$L(f, P) + L(f, Q) = L(f, P \cup Q) \leq \int_a^b f. \tag{7.4.27}$$

Thus

$$L(f, P) \leq \int_a^b f - L(f, Q), \tag{7.4.28}$$

so

$$\int_a^c f = \underline{\int}_a^c f \leq \int_a^b f - L(f,Q). \tag{7.4.29}$$

Hence

$$L(f,Q) \leq \int_a^b f - \int_a^c f, \tag{7.4.30}$$

so

$$\int_c^b f = \underline{\int}_c^b f \leq \int_a^b f - \int_a^c f. \tag{7.4.31}$$

Thus

$$\int_a^c f + \int_c^b f \leq \int_a^b f. \tag{7.4.32}$$

Hence

$$\int_a^c f + \int_c^b f = \int_a^b f. \tag{7.4.33}$$

Q.E.D.

Exercise 7.4.6. Suppose $f : [a,b] \to \mathbb{R}$ is bounded and B is a finite subset of (a,b). Show that if f is continuous on $[a,b] \setminus B$, then f is integrable on $[a,b]$.

Proposition 7.4.6. If f is integrable on $[a,b]$ with $f(x) \geq 0$ for all $x \in [a,b]$, then

$$\int_a^b f \geq 0. \tag{7.4.34}$$

Proof. The result follows from the fact that $L(f,P) \geq 0$ for any partition P of $[a,b]$. **Q.E.D.**

Proposition 7.4.7. Suppose f and g are both integrable on $[a,b]$. If, for all $x \in [a,b]$, $f(x) \leq g(x)$, then

$$\int_a^b f \leq \int_a^b g. \tag{7.4.35}$$

Proof. Since $g(x) - f(x) \geq 0$ for all $x \in [a,b]$, we have, using Propositions 7.4.2, 7.4.3, and 7.4.6,

$$\int_a^b g - \int_a^b f = \int_a^b (g-f) \geq 0. \tag{7.4.36}$$

Q.E.D.

Proposition 7.4.8. Suppose f is integrable on $[a,b]$, $m \in \mathbb{R}$, $M \in \mathbb{R}$, and $m \leq f(x) \leq M$ for all $x \in [a,b]$. Then

$$m(b-a) \leq \int_a^b f \leq M(b-a). \tag{7.4.37}$$

Proof. It follows from the previous proposition that

$$m(b-a) = \int_a^b m dx \leq \int_a^b f(x)dx \leq \int_a^b M dx = M(b-a). \tag{7.4.38}$$

Q.E.D.

Exercise 7.4.7. Show that

$$1 \leq \int_{-1}^{1} \frac{1}{1+x^2} dx \leq 2.$$

Exercise 7.4.8. Suppose f is continuous on $[0,1]$, differentiable on $(0,1)$, $f(0) = 0$, and $|f'(x)| \leq 1$ for all $x \in (0,1)$. Show that

$$-\frac{1}{2} \leq \int_0^1 f \leq \frac{1}{2}.$$

Exercise 7.4.9. Suppose f is integrable on $[a,b]$ and define $F : (a,b) \to \mathbb{R}$ by

$$F(x) = \int_a^x f.$$

Show that there exists $\alpha \in \mathbb{R}$ such that for any $x, y \in (a,b)$ with $x < y$,

$$|F(y) - F(x)| \leq \alpha(y - x).$$

Proposition 7.4.9. Suppose g is integrable on $[a,b]$, $g([a,b]) \subset [c,d]$, and $f : [c,d] \to \mathbb{R}$ is continuous. If $h = f \circ g$, then h is integrable on $[a,b]$.

Proof. Let $\epsilon > 0$ be given. Let

$$K > \sup\{f(x) : x \in [c,d]\} - \inf\{f(x) : x \in [c,d]\} \tag{7.4.39}$$

and choose $\delta > 0$ so that $\delta < \epsilon$ and

$$|f(x) - f(y)| < \frac{\epsilon}{2(b-a)} \tag{7.4.40}$$

whenever $|x - y| < \delta$. Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a,b]$ such that

$$U(g,P) - L(g,P) < \frac{\delta^2}{2K}. \tag{7.4.41}$$

For $i = 1, 2, \ldots, n$, let

$$m_i = \inf\{g(x) : x_{i-1} \leq x \leq x_i\}, \tag{7.4.42}$$

$$M_i = \sup\{g(x) : x_{i-1} \leq x \leq x_i\}, \tag{7.4.43}$$

$$w_i = \inf\{h(x) : x_{i-1} \leq x \leq x_i\}, \tag{7.4.44}$$

and

$$W_i = \sup\{h(x) : x_{i-1} \leq x \leq x_i\}. \tag{7.4.45}$$

Finally, let

$$I = \{i : i \in \mathbb{Z}, 1 \leq i \leq n, M_i - m_i < \delta\} \tag{7.4.46}$$

and

$$J = \{i : i \in \mathbb{Z}, 1 \leq i \leq n, M_i - m_i \geq \delta\}. \tag{7.4.47}$$

Note that

$$\begin{aligned} \delta \sum_{i \in J} (x_i - x_{i-1}) &\leq \sum_{i \in J} (M_i - m_i)(x_i - x_{i-1}) \\ &\leq \sum_{i=1}^{n} (M_i - m_i)(x_i - x_{i-1}) \\ &< \frac{\delta^2}{2K}, \end{aligned} \tag{7.4.48}$$

from which it follows that

$$\sum_{i \in J} (x_i - x_{i-1}) < \frac{\delta}{2K}. \tag{7.4.49}$$

Then

$$\begin{aligned} U(h,P) - L(h,P) &= \sum_{i \in I} (W_i - w_i)(x_i - x_{i-1}) + \sum_{i \in J} (W_i - w_i)(x_i - x_{i-1}) \\ &< \frac{\epsilon}{2(b-a)} \sum_{i \in I} (x_i - x_{i-1}) + K \sum_{i \in J} (x_i - x_{i-1}) \\ &< \frac{\epsilon}{2} + \frac{\delta}{2} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned} \tag{7.4.50}$$

Thus h is integrable on $[a,b]$. **Q.E.D.**

Proposition 7.4.10. Suppose f and g are both integrable on $[a,b]$. Then fg is integrable on $[a,b]$.

Proof. Since f and g are both integrable, both $f+g$ and $f-g$ are integrable. Hence, by the previous proposition, both $(f+g)^2$ and $(f-g)^2$ are integrable. Thus

$$\frac{1}{4}\left((f+g)^2 - (f-g)^2\right) = fg \tag{7.4.51}$$

is integrable on $[a,b]$. **Q.E.D.**

Proposition 7.4.11. Suppose f is integrable on $[a,b]$. Then $|f|$ is integrable on $[a,b]$ and

$$\left|\int_a^b f\right| \le \int_a^b |f|. \tag{7.4.52}$$

Proof. The integrability of $|f|$ follows from the integrability of f, the continuity of $g(x) = |x|$, and Proposition 7.4.9. For the inequality, note that

$$-|f(x)| \le f(x) \le |f(x)| \tag{7.4.53}$$

for all $x \in [a,b]$. Hence

$$-\int_a^b |f| \le \int_a^b f \le \int_a^b |f|, \tag{7.4.54}$$

from which the result follows. **Q.E.D.**

Exercise 7.4.10. Either prove the following statement or show it is false by finding a counterexample: If $f : [0,1] \to \mathbb{R}$ is bounded and f^2 is integrable on $[0,1]$, then f is integrable on $[0,1]$.

7.4.1 Extended definitions

Definition 7.4.1. If f is integrable on $[a,b]$, then we define

$$\int_b^a f = -\int_a^b f. \tag{7.4.55}$$

Moreover, if f is a function defined at a point $a \in \mathbb{R}$, we define

$$\int_a^a f = 0. \tag{7.4.56}$$

Exercise 7.4.11. Suppose f is integrable on a closed interval containing the points a, b, and c. Show that

$$\int_a^b f = \int_a^c f + \int_c^b f. \tag{7.4.57}$$

7.5 The Fundamental Theorem of Calculus

Theorem 7.5.1 (Fundamental Theorem of Calculus)**.** Suppose f is integrable on $[a, b]$. If F is continuous on $[a, b]$ and differentiable on (a, b) with $F'(x) = f(x)$ for all $x \in (a, b)$, then

$$\int_a^b f = F(b) - F(a). \tag{7.5.1}$$

Proof. Given $\epsilon > 0$, let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a, b]$ for which

$$U(f, P) - L(f, P) < \epsilon. \tag{7.5.2}$$

For $i = 1, 2, \ldots, n$, let $t_i \in (x_{i-1}, x_i)$ be points for which

$$F(x_i) - F(x_{i-1}) = f(t_i)(x_i - x_{i-1}). \tag{7.5.3}$$

Then

$$\sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) = \sum_{i=1}^{n} (F(x_i) - F(x_{i-1})) = F(b) - F(a). \tag{7.5.4}$$

But

$$L(f, P) \leq \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) \leq U(f, P), \tag{7.5.5}$$

so

$$\left| F(b) - F(a) - \int_a^b f \right| < \epsilon. \tag{7.5.6}$$

Since ϵ was arbitrary, we conclude that

$$\int_a^b f = F(b) - F(a). \tag{7.5.7}$$

Q.E.D.

Proposition 7.5.2 (Integration by parts)**.** Suppose f and g are integrable on $[a, b]$. If F and G are continuous on $[a, b]$ and differentiable on (a, b) with $F'(x) = f(x)$ and $G'(x) = g(x)$ for all $x \in (a, b)$, then

$$\int_a^b F(x)g(x)dx = F(b)G(b) - F(a)G(a) - \int_a^b f(x)G(x)dx. \tag{7.5.8}$$

Proof. By the Fundamental Theorem of Calculus,

$$\int_a^b (F(x)g(x) + f(x)G(x))dx = F(b)G(b) - F(a)G(a). \tag{7.5.9}$$

Q.E.D.

7.5.1 The other Fundamental Theorem of Calculus

Proposition 7.5.3. Suppose f is integrable on $[a,b]$ and $F:[a,b]\to\mathbb{R}$ is defined by

$$F(x)=\int_a^x f(t)dt. \tag{7.5.10}$$

Then F is uniformly continuous on $[a,b]$.

Proof. Let $\epsilon>0$ be given and let $M>0$ be such that $|f(x)|\le M$ for all $x\in[a,b]$. Then for any $x,y\in[a,b]$ with $x<y$ and $y-x<\frac{\epsilon}{M}$,

$$|F(y)-F(x)|=\left|\int_x^y f(t)dt\right|\le M(y-x)<\epsilon. \tag{7.5.11}$$

Hence F is uniformly continuous on $[a,b]$. **Q.E.D.**

The following theorem is often considered to be part of the Fundamental Theorem of Calculus.

Theorem 7.5.4. Suppose f is integrable on $[a,b]$ and continuous at $u\in(a,b)$. If $F:[a,b]\to\mathbb{R}$ is defined by

$$F(x)=\int_a^x f(t)dt, \tag{7.5.12}$$

then F is differentiable at u and $F'(u)=f(u)$.

Proof. Let $\epsilon>0$ be given and choose $\delta>0$ such that $|f(x)-f(u)|<\epsilon$ whenever $|x-u|<\delta$. Then if $0<h<\delta$, we have

$$\begin{aligned}\left|\frac{F(u+h)-F(u)}{h}-f(u)\right| &= \left|\frac{1}{h}\int_u^{u+h} f(t)dt-f(u)\right| \\ &= \left|\frac{1}{h}\int_u^{u+h}(f(t)-f(u))dt\right| \\ &< \epsilon. \end{aligned} \tag{7.5.13}$$

If $-\delta<h<0$, then

$$\begin{aligned}\left|\frac{F(u+h)-F(u)}{h}-f(u)\right| &= \left|-\frac{1}{h}\int_{u+h}^{u} f(t)dt-f(u)\right| \\ &= \left|\frac{1}{h}\int_{u+h}^{u} f(t)dt+f(u)\right| \\ &= \left|\frac{1}{h}\int_{u+h}^{u} f(t)dt-\frac{1}{h}\int_{u+h}^{u} f(u)dt\right| \\ &= \left|\frac{1}{h}\int_{u+h}^{u}(f(t)-f(u))dt\right| \\ &< \epsilon. \end{aligned} \tag{7.5.14}$$

Hence

$$F'(u) = \lim_{h \to 0} \frac{F(u+h) - F(u)}{h} = f(u). \qquad (7.5.15)$$

Q.E.D.

Proposition 7.5.5. If f is continuous on $[a, b]$, then there exists a function $F : [a, b] \to \mathbb{R}$ which is continuous on $[a, b]$ with $F'(x) = f(x)$ for all $x \in (a, b)$.

Proof. Let

$$F(x) = \int_a^x f(t)dt. \qquad (7.5.16)$$

Q.E.D.

Example 7.5.1. If

$$g(x) = \int_0^x \sqrt{1 + t^4}\ dt,$$

then $g'(x) = \sqrt{1 + x^4}$.

Proposition 7.5.6 (Integration by substitution)**.** Suppose I is an open interval, $\varphi : I \to \mathbb{R}$, $[a, b] \subset I$, and φ' is continuous on $[a, b]$. If $f : \varphi([a, b]) \to \mathbb{R}$ is continuous, then

$$\int_{\varphi(a)}^{\varphi(b)} f(u)du = \int_a^b f(\varphi(x))\varphi'(x)dx. \qquad (7.5.17)$$

Proof. Suppose $\varphi(a) \le \varphi(b)$ and let F be a function which is continuous on $[\varphi(a), \varphi(b)]$ with $F'(u) = f(u)$ for every $u \in (\varphi(a), \varphi(b))$. Let $g = F \circ \varphi$. Then

$$g'(x) = F'(\varphi(x))\varphi'(x) = f(\varphi(x))\varphi'(x), \qquad (7.5.18)$$

so

$$\begin{aligned} \int_a^b f(\varphi(x))\varphi'(x)dx &= g(b) - g(a) \\ &= F(\varphi(b)) - F(\varphi(a)) \\ &= \int_{\varphi(a)}^{\varphi(b)} f(u)du. \qquad (7.5.19) \end{aligned}$$

If $\varphi(a) > \varphi(b)$, then

$$\begin{aligned} \int_{\varphi(a)}^{\varphi(b)} f(u)du &= -\int_{\varphi(b)}^{\varphi(a)} f(u)du \\ &= -(F(\varphi(a)) - F(\varphi(b))) \\ &= F(\varphi(b)) - F(\varphi(a)) \\ &= g(b) - g(a) \\ &= \int_a^b f(\varphi(x))\varphi'(x)dx. \qquad (7.5.20) \end{aligned}$$

Q.E.D.

Exercise 7.5.1. Evaluate
$$\int_0^1 u\sqrt{u+1}\,du$$
using (a) integration by parts and (b) substitution.

Exercise 7.5.2. Suppose $\varphi : \mathbb{R} \to \mathbb{R}$ is differentiable on $\mathbb{R}$ and periodic with period 1 (that is, $\varphi(x+1) = \varphi(x)$ for every $x \in \mathbb{R}$). Show that for any continuous function $f : \mathbb{R} \to \mathbb{R}$,
$$\int_0^1 f(\varphi(x))\varphi'(x)dx = 0.$$

Theorem 7.5.7 (Integral Mean Value Theorem)**.** If f is continuous on $[a, b]$, then there exists $c \in [a, b]$ such that
$$\int_a^b f = f(c)(b-a). \tag{7.5.21}$$

Exercise 7.5.3. Prove the Integral Mean Value Theorem.

Theorem 7.5.8 (Generalized Integral Mean Value Theorem)**.** If f and g are continuous on $[a, b]$ and $g(x) > 0$ for all $x \in [a, b]$, then there exists $c \in [a, b]$ such that
$$\int_a^b fg = f(c)\int_a^b g. \tag{7.5.22}$$

Exercise 7.5.4. Prove the Generalized Integral Mean Value Theorem.

7.6 Taylor's theorem revisited

The following is a version of Taylor's Theorem with an alternative form of the remainder term.

Theorem 7.6.1 (Taylor's Theorem)**.** Suppose $f \in C^{(n+1)}(a, b)$, $\alpha \in (a, b)$, and
$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(\alpha)}{k!}(x-\alpha)^k. \tag{7.6.1}$$
Then, for any $x \in (a, b)$,
$$f(x) = P_n(x) + \int_\alpha^x \frac{f^{(n+1)}(t)}{n!}(x-t)^n dt. \tag{7.6.2}$$

Proof. By the Fundamental Theorem of Calculus, we have

$$\int_\alpha^x f'(t)dt = f(x) - f(\alpha), \tag{7.6.3}$$

which implies that

$$f(x) = f(\alpha) + \int_\alpha^x f'(t)dt. \tag{7.6.4}$$

Hence the theorem holds for $n = 0$. Suppose the result holds for $n = k-1$, that is,

$$f(x) = P_{k-1}(x) + \int_\alpha^x \frac{f^{(k)}(t)}{(k-1)!}(x-t)^{k-1}dt. \tag{7.6.5}$$

Let

$$F(t) = f^{(k)}(t), \tag{7.6.6}$$

$$g(t) = \frac{(x-t)^{k-1}}{(k-1)!}, \tag{7.6.7}$$

and

$$G(t) = -\frac{(x-t)^k}{k!}. \tag{7.6.8}$$

Then

$$\begin{aligned}\int_\alpha^x \frac{f^{(k)}(t)}{(k-1)!}(x-t)^{k-1}dt &= \int_\alpha^x F(t)g(t)dt \\ &= F(x)G(x) - F(\alpha)G(\alpha) - \int_\alpha^x F'(t)G(t)dt \\ &= \frac{f^{(k)}(\alpha)(x-\alpha)^k}{k!} + \int_\alpha^x \frac{f^{(k+1)}(t)}{k!}(x-t)^k dt. \end{aligned} \tag{7.6.9}$$

Hence

$$f(x) = P_k(x) + \int_\alpha^x \frac{f^{(k+1)}(t)}{k!}(x-t)^k dt, \tag{7.6.10}$$

and so the theorem holds for $n = k$. **Q.E.D.**

Exercise 7.6.1 (Cauchy form of the remainder)**.** Under the conditions of Taylor's Theorem as just stated, show that

$$\int_\alpha^x \frac{f^{(n+1)}(t)}{n!}(x-t)^n dt = \frac{f^{(n+1)}(\gamma)}{n!}(x-\gamma)^n(x-\alpha) \tag{7.6.11}$$

for some γ between α and x.

Exercise 7.6.2 (Lagrange form of the remainder)**.** Under the conditions of Taylor's Theorem as just stated, show that

$$\int_{\alpha}^{x} \frac{f^{(n+1)}(t)}{n!}(x-t)^n dt = \frac{f^{(n+1)}(\gamma)}{(n+1)!}(x-\alpha)^{n+1} \tag{7.6.12}$$

for some γ between α and x. Note that this is the form of the remainder in Theorem 6.6.1, although under slightly more restrictive assumptions.

7.7 An improper integral

Definition 7.7.1. If f is integrable on $[a, b]$ for all $b > a$ and

$$\lim_{b \to +\infty} \int_a^b f(x)dx \tag{7.7.1}$$

exists, then we define

$$\int_a^{+\infty} f(x)dx = \lim_{b \to +\infty} \int_a^b f(x)dx. \tag{7.7.2}$$

If f is integrable on $[a, b]$ for all $a < b$ and

$$\lim_{a \to -\infty} \int_a^b f(x)dx \tag{7.7.3}$$

exists, then we define

$$\int_{-\infty}^{b} f(x)dx = \lim_{a \to -\infty} \int_a^b f(x)dx. \tag{7.7.4}$$

Both of these integrals are examples of *improper* integrals.

Proposition 7.7.1. Suppose f is continuous on $[a, \infty)$ and $f(x) \geq 0$ for all $x \geq a$. If there exists $g : [a, +\infty) \to \mathbb{R}$ for which

$$\int_a^{+\infty} g(x)dx \tag{7.7.5}$$

exists and $g(x) \geq f(x)$ for all $x \geq a$, then

$$\int_a^{+\infty} f(x)dx \tag{7.7.6}$$

exists.

Exercise 7.7.1. Prove the preceding proposition.

Example 7.7.1. Suppose
$$f(x) = \frac{1}{1+x^2}$$
and
$$g(x) = \begin{cases} 1, & \text{if } 0 \leq x < 1, \\ \frac{1}{x^2}, & \text{if } x \geq 1. \end{cases}$$
Then, for $b > 1$,
$$\int_0^b g(x)dx = \int_0^1 dx + \int_1^b \frac{1}{x^2}\, dx = 1 + 1 - \frac{1}{b} = 2 - \frac{1}{b},$$
so
$$\int_0^{+\infty} g(x)dx = \lim_{b\to\infty} \left(2 - \frac{1}{b}\right) = 2.$$
Since $0 < f(x) \leq g(x)$ for all $x \geq 0$, it follows that
$$\int_0^{+\infty} \frac{1}{1+x^2}\, dx$$
exists, and, moreover,
$$\int_0^{+\infty} \frac{1}{1+x^2} dx < 2.$$
Also, the substitution $u = -x$ shows that
$$\int_{-\infty}^0 \frac{1}{1+x^2}\, dx = -\int_{+\infty}^0 \frac{1}{1+u^2}\, du = \int_0^{+\infty} \frac{1}{1+u^2}\, du.$$

Chapter 8

More Functions

8.1 The arctangent function

Definition 8.1.1. For any $x \in \mathbb{R}$, we call

$$\arctan(x) = \int_0^x \frac{1}{1+t^2}\, dt \tag{8.1.1}$$

the *arctangent* of x.

Proposition 8.1.1. The arctangent function is differentiable at every $x \in \mathbb{R}$. Moreover, if $f(x) = \arctan(x)$, then

$$f'(x) = \frac{1}{1+x^2}. \tag{8.1.2}$$

Proof. The result follows immediately from Theorem 7.5.4. **Q.E.D.**

Proposition 8.1.2. The arctangent is increasing on $\mathbb{R}$.

Proof. The result follows immediately from the previous proposition and the fact that

$$\frac{1}{1+x^2} > 0 \tag{8.1.3}$$

for every $x \in \mathbb{R}$. **Q.E.D.**

Definition 8.1.2. $\pi = 2 \lim_{x \to +\infty} \arctan(x) = 2 \int_0^{+\infty} \frac{1}{1+t^2}\, dt.$

Note that $0 < \pi < 4$ by Example 7.7.1.

The following proposition says that the arctangent function is an *odd* function.

Proposition 8.1.3. For any $x \in \mathbb{R}$, $\arctan(x) = -\arctan(-x)$.

Proof. Using the substitution $t = -u$, we have

$$\arctan(x) = \int_0^x \frac{1}{1+t^2}\,dt = -\int_0^{-x} \frac{1}{1+u^2}\,du = -\arctan(-x). \tag{8.1.4}$$

Q.E.D.

It now follows that

$$\lim_{x\to-\infty} \arctan(x) = -\lim_{x\to-\infty} \arctan(-x) = -\frac{\pi}{2}. \tag{8.1.5}$$

Hence the range of the arctangent function is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Proposition 8.1.4. If $x > 0$, then

$$\arctan(x) + \arctan\left(\frac{1}{x}\right) = \frac{\pi}{2}. \tag{8.1.6}$$

Proof. Using the substitution $t = \frac{1}{u}$, we have

$$\begin{aligned}
\arctan\left(\frac{1}{x}\right) &= \int_0^{\frac{1}{x}} \frac{1}{1+t^2}\,dt \\
&= \int_{+\infty}^{x} \frac{1}{1+\frac{1}{u^2}}\left(-\frac{1}{u^2}\right) du \\
&= -\int_{+\infty}^{x} \frac{1}{1+u^2}\,du \\
&= \int_{x}^{+\infty} \frac{1}{1+u^2}\,du \\
&= \frac{\pi}{2} - \int_0^{x} \frac{1}{1+u^2}\,du \\
&= \frac{\pi}{2} - \arctan(x).
\end{aligned} \tag{8.1.7}$$

Q.E.D.

Proposition 8.1.5. If $x < 0$, then

$$\arctan(x) + \arctan\left(\frac{1}{x}\right) = -\frac{\pi}{2}. \tag{8.1.8}$$

Proof. The result follows immediately from the preceding proposition and the fact that arctangent is an odd function. **Q.E.D.**

Exercise 8.1.1. Show that $\arctan(1) = \frac{\pi}{4}$ and $\arctan(-1) = -\frac{\pi}{4}$.

8.2 The tangent function

Let

$$A = \left\{ \frac{\pi}{2} + n\pi : n \in \mathbb{Z} \right\} \tag{8.2.1}$$

and $D = \mathbb{R} \setminus A$. Let

$$t : \left(-\frac{\pi}{2}, \frac{\pi}{2} \right) \to \mathbb{R} \tag{8.2.2}$$

be the inverse of the arctangent function. Note that t is increasing and differentiable on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. We may extend t to a function on D as follows: For any $x \in D$, let

$$g(x) = \sup \left\{ n : n \in \mathbb{Z}, -\frac{\pi}{2} + n\pi < x \right\} \tag{8.2.3}$$

and define $T(x) = t(x - g(x)\pi)$.

Definition 8.2.1. With the notation of the above discussion, for any $x \in D$, we call the value $T(x)$ the *tangent* of x, which we denote $\tan(x)$.

Proposition 8.2.1. The tangent function has domain D (as defined above), range $\mathbb{R}$, and is differentiable at every point $x \in D$. Moreover, the tangent function is increasing on each interval of the form

$$\left(-\frac{\pi}{2} + n\pi, \frac{\pi}{2} + n\pi \right), \tag{8.2.4}$$

$n \in \mathbb{Z}$, with

$$\tan\left(\left(\frac{\pi}{2} + n\pi \right) + \right) = -\infty \tag{8.2.5}$$

and

$$\tan\left(\left(\frac{\pi}{2} + n\pi \right) - \right) = +\infty. \tag{8.2.6}$$

Proof. These results follow immediately from our definitions. **Q.E.D.**

Definition 8.2.2. Let $E \subset \mathbb{R}$. We say a function $f : E \to \mathbb{R}$ is *periodic* if there exists a real number $p > 0$ such that, for each $x \in E$, $x + p \in E$ and $f(x + p) = f(x)$. We say p is the *period* of a periodic function f if p is the smallest positive number for which $f(x + p) = f(x)$ for all $x \in E$.

Proposition 8.2.2. The tangent function has period π.

Proof. The result follows immediately from our definitions. **Q.E.D.**

Proposition 8.2.3 (Addition formula for tangent)**.** For any $x, y \in D$ with $x + y \in D$,

$$\tan(x + y) = \frac{\tan(x) + \tan(y)}{1 - \tan(x)\tan(y)}. \tag{8.2.7}$$

Proof. Suppose $y_1, y_2 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ with $y_1 + y_2 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Let $x_1 = \tan(y_1)$ and $x_2 = \tan(y_2)$. Note that if $x_1 > 0$, then $x_1 x_2 \geq 1$ would imply that

$$x_2 \geq \frac{1}{x_1}, \tag{8.2.8}$$

which in turn implies that

$$\begin{aligned} y_1 + y_2 &= \arctan(x_1) + \arctan(x_2) \\ &\geq \arctan(x_1) + \arctan\left(\frac{1}{x_1}\right) \\ &= \frac{\pi}{2}, \end{aligned} \tag{8.2.9}$$

contrary to our assumptions. Similarly, if $x_1 < 0$, then $x_1 x_2 \geq 1$ would imply that

$$x_2 \leq \frac{1}{x_1}, \tag{8.2.10}$$

which in turn implies that

$$\begin{aligned} y_1 + y_2 &= \arctan(x_1) + \arctan(x_2) \\ &\leq \arctan(x_1) + \arctan\left(\frac{1}{x_1}\right) \\ &= -\frac{\pi}{2}, \end{aligned} \tag{8.2.11}$$

contrary to our assumptions. Thus we must have $x_1 x_2 < 1$. Moreover, suppose u is a number between $-x_1$ and x_2. If $x_1 > 0$, then

$$x_2 < \frac{1}{x_1}, \tag{8.2.12}$$

and so

$$u < \frac{1}{x_1}. \tag{8.2.13}$$

If $x_1 < 0$, then

$$x_2 > \frac{1}{x_1}, \tag{8.2.14}$$

and so

$$u > \frac{1}{x_1}. \tag{8.2.15}$$

Now let

$$x = \frac{x_1 + x_2}{1 - x_1 x_2}. \tag{8.2.16}$$

We want to show that

$$\arctan(x) = \arctan(x_1) + \arctan(x_2), \tag{8.2.17}$$

which will imply that

$$\frac{\tan(y_1)+\tan(y_2)}{1-\tan(y_1)\tan(y_2)} = \tan(y_1+y_2). \tag{8.2.18}$$

We need to compute

$$\arctan(x) = \arctan\left(\frac{x_1+x_2}{1-x_1x_2}\right) = \int_0^{\frac{x_1+x_2}{1-x_1x_2}} \frac{1}{1+t^2}\,dt. \tag{8.2.19}$$

Let

$$t = \varphi(u) = \frac{x_1+u}{1-x_1u}, \tag{8.2.20}$$

where u varies between $-x_1$, where $t=0$, and x_2, where $t=x$. Now

$$\varphi'(u) = \frac{(1-x_1u)-(x_1+u)(-x_1)}{(1-x_1u)^2} = \frac{1+x_1^2}{(1-x_1u)^2}, \tag{8.2.21}$$

which is always positive, thus showing that φ is an increasing function, and

$$\begin{aligned}\frac{1}{1+t^2} &= \frac{1}{1+\left(\dfrac{x_1+u}{1-x_1u}\right)^2}\\ &= \frac{(1-x_1u)^2}{(1-x_1u)^2+(x_1+u)^2}\\ &= \frac{(1-x_1u)^2}{(1+x_1^2)(1+u^2)}.\end{aligned} \tag{8.2.22}$$

Hence

$$\begin{aligned}\arctan(x) &= \int_{-x_1}^{x_2} \frac{1}{1+u^2}\,du\\ &= \int_{-x_1}^{0} \frac{1}{1+u^2}\,du + \int_0^{x_2} \frac{1}{1+u^2}\,du\\ &= -\int_0^{-x_1} \frac{1}{1+u^2}\,du + \arctan(x_2)\\ &= -\arctan(-x_1)+\arctan(x_2)\\ &= \arctan(x_1)+\arctan(x_2).\end{aligned} \tag{8.2.23}$$

Now suppose $y_1, y_2 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ with $y_1+y_2 > \frac{\pi}{2}$. Then $y_1+y_2 \in \left(\frac{\pi}{2}, \pi\right)$, $x_1 > 0$, $x_2 > 0$, and

$$x_2 > \frac{1}{x_1}. \tag{8.2.24}$$

With u and x as above, note then that as u increases from $-x_1$ to $\frac{1}{x_1}$, t increases from 0 to $+\infty$, and as u increases from $\frac{1}{x_1}$ to x_2, t increases from $-\infty$ to x.

Hence we have

$$\begin{aligned}
\arctan(x) + \pi &= \int_0^x \frac{1}{1+t^2}\,dt + \int_{-\infty}^0 \frac{1}{1+t^2}\,dt + \int_0^{+\infty} \frac{1}{1+t^2}\,dt \\
&= \int_{-\infty}^x \frac{1}{1+t^2}\,dt + \int_0^{+\infty} \frac{1}{1+t^2}\,dt \\
&= \int_{\frac{1}{x_1}}^{x_2} \frac{1}{1+u^2}\,du + \int_{-x_1}^{\frac{1}{x_1}} \frac{1}{1+u^2}\,du \\
&= \int_{-x_1}^{x_2} \frac{1}{1+u^2}\,du \\
&= \arctan(x_2) - \arctan(-x_1) \\
&= \arctan(x_2) + \arctan(x_1). \qquad (8.2.25)
\end{aligned}$$

Hence

$$\begin{aligned}
\tan(y_1 + y_2) &= \tan(y_1 + y_2 - \pi) \\
&= \tan(\arctan(x)) \\
&= \frac{x_1 + x_2}{1 - x_1 x_2} \\
&= \frac{\tan(y_1) + \tan(y_2)}{1 - \tan(y_1)\tan(y_2)}. \qquad (8.2.26)
\end{aligned}$$

The case when $x_1 < 0$ may be handled similarly; it then follows that the addition formula holds for all $y_1, y_2 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. The case for arbitrary $y_1, y_2 \in D$ with $y_1 + y_2 \in D$ then follows from the periodicity of the tangent function. **Q.E.D.**

8.3 The sine and cosine Functions

We begin by defining functions

$$s : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right] \to \mathbb{R} \qquad (8.3.1)$$

and

$$c : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right] \to \mathbb{R} \qquad (8.3.2)$$

by

$$s(x) = \begin{cases} \dfrac{\tan(x)}{\sqrt{1+\tan^2(x)}}, & \text{if } x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \\ 1, & \text{if } x = \dfrac{\pi}{2} \end{cases} \qquad (8.3.3)$$

and

$$c(x) = \begin{cases} \dfrac{1}{\sqrt{1+\tan^2(x)}}, & \text{if } x \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right), \\ 0, & \text{if } x = \dfrac{\pi}{2}. \end{cases} \qquad (8.3.4)$$

Note that

$$\lim_{x \to \frac{\pi}{2}^-} s(x) = \lim_{y \to +\infty} \frac{y}{\sqrt{1+y^2}} = \lim_{y \to +\infty} \frac{1}{\sqrt{1+\frac{1}{y^2}}} = 1 \tag{8.3.5}$$

and

$$\lim_{x \to \frac{\pi}{2}^-} c(x) = \lim_{y \to +\infty} \frac{1}{\sqrt{1+y^2}} = \lim_{y \to +\infty} \frac{\frac{1}{y}}{\sqrt{1+\frac{1}{y^2}}} = 0, \tag{8.3.6}$$

which shows that both s and c are continuous functions.

Next, we extend the definitions of s and c to functions

$$S : \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right] \to \mathbb{R} \tag{8.3.7}$$

and

$$C : \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right] \to \mathbb{R} \tag{8.3.8}$$

by defining

$$S(x) = \begin{cases} s(x), & \text{if } x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right], \\ -s(x-\pi), & \text{if } x \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right] \end{cases} \tag{8.3.9}$$

and

$$C(x) = \begin{cases} c(x), & \text{if } x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right], \\ -c(x-\pi), & \text{if } x \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right]. \end{cases} \tag{8.3.10}$$

Note that

$$\lim_{x \to \frac{\pi}{2}^+} S(x) = \lim_{x \to -\frac{\pi}{2}^+} -s(x) = -\lim_{y \to -\infty} \frac{y}{\sqrt{1+y^2}} = -\lim_{y \to -\infty} \frac{1}{-\sqrt{1+\frac{1}{y^2}}} = 1 \tag{8.3.11}$$

and

$$\lim_{x \to \frac{\pi}{2}^+} C(x) = \lim_{x \to -\frac{\pi}{2}^+} -c(x) = -\lim_{y \to -\infty} \frac{1}{\sqrt{1+y^2}} = -\lim_{y \to -\infty} \frac{\frac{1}{y}}{-\sqrt{1+\frac{1}{y^2}}} = 0, \tag{8.3.12}$$

which shows that both S and C are continuous at $\frac{\pi}{2}$. Thus both S and C are continuous.

Finally, for any $x \in \mathbb{R}$, let

$$g(x) = \sup\left\{n : n \in \mathbb{Z}, -\frac{\pi}{2} + 2n\pi < x\right\} \tag{8.3.13}$$

and define

$$\sin(x) = S(x - 2\pi g(x)) \tag{8.3.14}$$

and

$$\cos(x) = C(x - 2\pi g(x)). \tag{8.3.15}$$

Definition 8.3.1. With the notation as above, for any $x \in \mathbb{R}$, we call $\sin(x)$ and $\cos(x)$ the *sine* and *cosine* of x, respectively.

Proposition 8.3.1. The sine and cosine functions are continuous on $\mathbb{R}$.

Proof. From the definitions, it is sufficient to verify continuity at $\frac{3\pi}{2}$. Now

$$\lim_{x \to \frac{3\pi}{2}^-} \sin(x) = \lim_{x \to \frac{3\pi}{2}^-} S(x) = S\left(\frac{3\pi}{2}\right) = -s\left(\frac{\pi}{2}\right) = -1 \tag{8.3.16}$$

and

$$\begin{aligned} \lim_{x \to \frac{3\pi}{2}^+} \sin(x) &= \lim_{x \to \frac{3\pi}{2}^+} S(x - 2\pi) \\ &= \lim_{x \to -\frac{\pi}{2}^+} s(x) \\ &= \lim_{y \to -\infty} \frac{y}{\sqrt{1 + y^2}} \\ &= \lim_{y \to -\infty} \frac{1}{-\sqrt{1 + \frac{1}{y^2}}} \\ &= -1, \end{aligned} \tag{8.3.17}$$

and so sine is continuous at $\frac{3\pi}{2}$. Similarly,

$$\lim_{x \to \frac{3\pi}{2}^-} \cos(x) = \lim_{x \to \frac{3\pi}{2}^-} C(x) = C\left(\frac{3\pi}{2}\right) = -c\left(\frac{\pi}{2}\right) = 0 \tag{8.3.18}$$

and

$$\begin{aligned} \lim_{x \to \frac{3\pi}{2}^+} \cos(x) &= \lim_{x \to \frac{3\pi}{2}^+} C(x - 2\pi) \\ &= \lim_{x \to -\frac{\pi}{2}^+} c(x) \\ &= \lim_{y \to -\infty} \frac{1}{\sqrt{1 + y^2}} \\ &= \lim_{y \to -\infty} \frac{\frac{1}{y}}{-\sqrt{1 + \frac{1}{y^2}}} \\ &= 0, \end{aligned} \tag{8.3.19}$$

and so cosine is continuous at $\frac{3\pi}{2}$. **Q.E.D.**

8.3.1 Properties of sine and cosine

Proposition 8.3.2. The sine and cosine functions are periodic with period 2π.

Proof. The result follows immediately from the definitions. **Q.E.D.**

Proposition 8.3.3. For any $x \in \mathbb{R}$, $\sin(-x) = -\sin(x)$ and $\cos(-x) = \cos(x)$.

Proof. The result follows immediately from the definitions. **Q.E.D.**

Proposition 8.3.4. For any $x \in \mathbb{R}$, $\sin^2(x) + \cos^2(x) = 1$.

Proof. The result follows immediately from the definition of s and c. **Q.E.D.**

Proposition 8.3.5. The range of both the sine and cosine functions is $[-1, 1]$.

Proof. The result follows immediately from the definitions along with the facts that

$$\sqrt{1+y^2} \geq \sqrt{y^2} = |y| \tag{8.3.20}$$

and

$$\sqrt{1+y^2} \geq 1 \tag{8.3.21}$$

for any $y \in \mathbb{R}$. **Q.E.D.**

Proposition 8.3.6. For any x in the domain of the tangent function,

$$\tan(x) = \frac{\sin(x)}{\cos(x)}. \tag{8.3.22}$$

Proof. The result follows immediately from the definitions. **Q.E.D.**

Proposition 8.3.7. For any x in the domain of the tangent function,

$$\sin^2(x) = \frac{\tan^2(x)}{1+\tan^2(x)} \tag{8.3.23}$$

and

$$\cos^2(x) = \frac{1}{1+\tan^2(x)}. \tag{8.3.24}$$

Proof. The result follows immediately from the definitions. **Q.E.D.**

Proposition 8.3.8. For any $x, y \in \mathbb{R}$,

$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y). \tag{8.3.25}$$

Proof. First suppose x, y, and $x+y$ are in the domain of the tangent function.

Then

$$\begin{aligned}
\cos^2(x+y) &= \frac{1}{1+\tan^2(x+y)} \\
&= \frac{1}{1+\left(\dfrac{\tan(x)+\tan(y)}{1-\tan(x)\tan(y)}\right)^2} \\
&= \frac{(1-\tan(x)\tan(y))^2}{(1-\tan(x)\tan(y))^2+(\tan(x)+\tan(y))^2} \\
&= \frac{(1-\tan(x)\tan(y))^2}{(1+\tan^2(x))(1+\tan^2(y))} \\
&= \left(\frac{1}{\sqrt{1+\tan^2(x)}\sqrt{1+\tan^2(y)}} - \frac{\tan(x)\tan(y)}{\sqrt{1+\tan^2(x)}\sqrt{1+\tan^2(y)}}\right)^2 \\
&= (\cos(x)\cos(y)-\sin(x)\sin(y))^2.
\end{aligned}$$

Hence

$$\cos(x+y) = \pm(\cos(x)\cos(y)-\sin(x)\sin(y)). \tag{8.3.26}$$

Consider a fixed value of x. Note that the positive sign must be chosen when $y = 0$. Moreover, increasing y by π changes the sign on both sides, so the positive sign must be chosen when y is any multiple of π. Since sine and cosine are continuous functions,the choice of sign could change only at points at which both sides are 0, but these points are separated by a distance of π, so we must always choose the positive sign. Hence we have

$$\cos(x+y) = \cos(x)\cos(y)-\sin(x)\sin(y) \tag{8.3.27}$$

for all $x, y \in \mathbb{R}$ for which x, y, and $x+y$ are in the domain of the tangent function. The identity for the other values of x and y now follows from the continuity of the sine and cosine functions. **Q.E.D.**

Proposition 8.3.9. For any $x, y \in \mathbb{R}$,

$$\sin(x+y) = \sin(x)\cos(y)+\sin(y)\cos(x). \tag{8.3.28}$$

Exercise 8.3.1. Prove the previous proposition.

Exercise 8.3.2. Show that for any $x \in \mathbb{R}$,

$$\sin\left(\frac{\pi}{2}-x\right) = \cos(x) \tag{8.3.29}$$

and

$$\cos\left(\frac{\pi}{2}-x\right) = \sin(x). \tag{8.3.30}$$

Exercise 8.3.3. Show that for any $x \in \mathbb{R}$,

$$\sin(2x) = 2\sin(x)\cos(x) \tag{8.3.31}$$

and

$$\cos(2x) = \cos^2(x) - \sin^2(x). \tag{8.3.32}$$

Exercise 8.3.4. Show that for any $x \in \mathbb{R}$,

$$\sin^2(x) = \frac{1-\cos(2x)}{2} \tag{8.3.33}$$

and

$$\cos^2(x) = \frac{1+\cos(2x)}{2}. \tag{8.3.34}$$

Exercise 8.3.5. Show that

$$\sin\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}, \tag{8.3.35}$$

$$\sin\left(\frac{\pi}{6}\right) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}, \tag{8.3.36}$$

and

$$\sin\left(\frac{\pi}{3}\right) = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}. \tag{8.3.37}$$

8.3.2 The calculus of the trigonometric functions

Proposition 8.3.10. $\displaystyle\lim_{x\to 0}\frac{\arctan(x)}{x} = 1.$

Proof. Using l'Hôpital's rule,

$$\lim_{x\to 0}\frac{\arctan(x)}{x} = \lim_{x\to 0}\frac{\frac{1}{1+x^2}}{1} = 1. \tag{8.3.38}$$

Q.E.D.

Proposition 8.3.11. $\displaystyle\lim_{x\to 0}\frac{\tan(x)}{x} = 1.$

Proof. Letting $x = \arctan(u)$, we have

$$\lim_{x\to 0}\frac{\tan(x)}{x} = \lim_{u\to 0}\frac{u}{\arctan(u)} = 1. \tag{8.3.39}$$

Q.E.D.

Proposition 8.3.12. $\displaystyle\lim_{x\to 0}\frac{\sin(x)}{x} = 1.$

Proof. We have

$$\lim_{x\to 0} \frac{\sin(x)}{x} = \lim_{x\to 0} \frac{\tan(x)}{x} \cos(x) = 1. \tag{8.3.40}$$

Q.E.D.

Proposition 8.3.13. $\lim_{x\to 0} \frac{1-\cos(x)}{x} = 0.$

Proof. We have

$$\begin{aligned}
\lim_{x\to 0} \frac{1-\cos(x)}{x} &= \lim_{x\to 0} \left(\frac{1-\cos(x)}{x}\right)\left(\frac{1+\cos(x)}{1+\cos(x)}\right) \\
&= \lim_{x\to 0} \frac{1-\cos^2(x)}{x(1+\cos(x))} \\
&= \lim_{x\to 0} \left(\frac{\sin(x)}{x}\right)\left(\frac{\sin(x)}{1+\cos(x)}\right) \\
&= (1)(0) \\
&= 0.
\end{aligned} \tag{8.3.41}$$

Q.E.D.

Proposition 8.3.14. If $f(x) = \sin(x)$, then $f'(x) = \cos(x)$.

Proof. We have

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h} \\
&= \lim_{h\to 0} \frac{\sin(x)\cos(h) + \sin(h)\cos(x) - \sin(x)}{h} \\
&= \sin(x) \lim_{h\to 0} \frac{\cos(h) - 1}{h} + \cos(x) \lim_{h\to 0} \frac{\sin(h)}{h} \\
&= \cos(x).
\end{aligned} \tag{8.3.42}$$

Q.E.D.

Proposition 8.3.15. If $f(x) = \cos(x)$, then $f'(x) = -\sin(x)$.

Exercise 8.3.6. Prove the previous proposition.

Definition 8.3.2. For appropriate $x \in \mathbb{R}$, we call

$$\cot(x) = \frac{\cos x}{\sin(x)}, \tag{8.3.43}$$

$$\sec(x) = \frac{1}{\cos(x)}, \tag{8.3.44}$$

and

$$\csc(x) = \frac{1}{\sin(x)} \tag{8.3.45}$$

the *cotangent*, *secant*, and *cosecant* of x, respectively.

Exercise 8.3.7. If $f(x) = \tan(x)$ and $g(x) = \cot(x)$, show that

$$f'(x) = \sec^2(x) \tag{8.3.46}$$

and

$$g'(x) = -\csc^2(x). \tag{8.3.47}$$

Exercise 8.3.8. If $f(x) = \sec(x)$ and $g(x) = \csc(x)$, show that

$$f'(x) = \sec(x)\tan(x), \tag{8.3.48}$$

and

$$g'(x) = -\csc(x)\cot(x). \tag{8.3.49}$$

Proposition 8.3.16. $2\int_{-1}^{1}\sqrt{1-x^2}dx = \pi.$

Proof. Let $x = \sin(u)$. Then as u varies from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, x varies from -1 to 1. And, for these values, we have

$$\sqrt{1-x^2} = \sqrt{1-\sin^2(u)} = \sqrt{\cos^2(u)} = |\cos(u)| = \cos(u). \tag{8.3.50}$$

Hence

$$\begin{aligned}
\int_{-1}^{1}\sqrt{1-x^2}dx &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2(u)du \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1+\cos(2u)}{2}\, du \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}\, du + \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(2u)du \\
&= \frac{\pi}{2} + \frac{1}{4}(\sin(\pi) - \sin(-\pi)) \\
&= \frac{\pi}{2}.
\end{aligned} \tag{8.3.51}$$

Q.E.D.

Exercise 8.3.9. Find the Taylor polynomial P_9 of order 9 for $f(x) = \sin(x)$ at 0. Note that this is equal to the Taylor polynomial of order 10 for f at 0. Is $P_9\left(\frac{1}{2}\right)$ an overestimate or an underestimate for $\sin\left(\frac{1}{2}\right)$? Find an upper bound for the error in this approximation.

8.4 The logarithm function

Definition 8.4.1. Given a positive real number x, we call

$$\log(x) = \int_1^x \frac{1}{t}\, dt \tag{8.4.1}$$

the *logarithm* of x.

Note that $\log(1) = 0$, $\log(x) < 0$ when $0 < x < 1$, and $\log(x) > 0$ when $x > 1$.

Proposition 8.4.1. The function $f(x) = \log(x)$ is an increasing, differentiable function with

$$f'(x) = \frac{1}{x} \tag{8.4.2}$$

for all $x > 0$.

Proof. Using the Fundamental Theorem of Calculus, we have

$$f'(x) = \frac{1}{x} > 0 \tag{8.4.3}$$

for all $x > 0$, from which the result follows. **Q.E.D.**

Proposition 8.4.2. For any $x > 0$,

$$\log\left(\frac{1}{x}\right) = -\log(x). \tag{8.4.4}$$

Proof. Using the substitution $t = \frac{1}{u}$, we have

$$\log\left(\frac{1}{x}\right) = \int_1^{\frac{1}{x}} \frac{1}{t}\, dt = \int_1^x u\left(-\frac{1}{u^2}\right)\, du = -\int_1^x \frac{1}{u}\, du = -\log(x). \tag{8.4.5}$$

Q.E.D.

Proposition 8.4.3. For any positive real numbers x and y,

$$\log(xy) = \log(x) + \log(y). \tag{8.4.6}$$

Proof. Using the substitution $t = xu$, we have

$$\begin{aligned} \log(xy) &= \int_1^{xy} \frac{1}{t}\, dt \\ &= \int_{\frac{1}{x}}^{y} \frac{x}{xu}\, du \\ &= \int_{\frac{1}{x}}^{1} \frac{1}{u}\, du + \int_1^y \frac{1}{u}\, du \\ &= -\int_1^{\frac{1}{x}} \frac{1}{u}\, du + \log(y) \\ &= -\log\left(\frac{1}{x}\right) + \log(y) \\ &= \log(x) + \log(y). \end{aligned} \tag{8.4.7}$$

Q.E.D.

Proposition 8.4.4. If $r \in \mathbb{Q}$ and x is a positive real number, then

$$\log(x^r) = r \log(x). \tag{8.4.8}$$

Proof. Using the substitution $t = u^r$, we have

$$\log(x^r) = \int_1^{x^r} \frac{1}{t}\, dt = \int_1^x \frac{ru^{r-1}}{u^r}\, du = r \int_1^x \frac{1}{u}\, du = r\log(x). \tag{8.4.9}$$

Q.E.D.

Proposition 8.4.5. $\lim_{x\to+\infty} \log(x) = +\infty$ and $\lim_{x\to 0^+} \log(x) = -\infty$.

Proof. Given a real number M, choose an integer n for which $n\log(2) > M$ (there exists such an n since $\log(2) > 0$). Then for any $x > 2^n$, we have

$$\log(x) > \log(2^n) = n\log(2) > M. \tag{8.4.10}$$

Hence $\lim_{x\to+\infty} \log(x) = +\infty$.

Similarly, given any real number M, we may choose an integer n for which $-n\log(2) < M$. Then for any $0 < x < \frac{1}{2^n}$, we have

$$\log(x) < \log\left(\frac{1}{2^n}\right) = -n\log(2) < M. \tag{8.4.11}$$

Hence $\lim_{x\to 0^+} \log(x) = -\infty$. **Q.E.D.**

Note that the logarithm function has domain $(0, +\infty)$ and range $(-\infty, +\infty)$.

Exercise 8.4.1. Show that for any rational number $\alpha > 0$,

$$\lim_{x\to+\infty} x^\alpha = +\infty.$$

Proposition 8.4.6. For any rational number $\alpha > 0$,

$$\lim_{x\to+\infty} \frac{\log(x)}{x^\alpha} = 0. \tag{8.4.12}$$

Proof. Choose a rational number β such that $0 < \beta < \alpha$. Now for any $t > 1$,

$$\frac{1}{t} < \frac{1}{t}\, t^\beta = \frac{1}{t^{1-\beta}}. \tag{8.4.13}$$

Hence

$$\log(x) = \int_1^x \frac{1}{t}\, dt < \int_1^x \frac{1}{t^{1-\beta}}\, dt = \frac{x^\beta - 1}{\beta} < \frac{x^\beta}{\beta} \tag{8.4.14}$$

whenever $x > 1$. Thus

$$0 < \frac{\log(x)}{x^{\alpha}} < \frac{1}{\beta x^{\alpha-\beta}} \tag{8.4.15}$$

for $x > 1$. But

$$\lim_{x\to+\infty} \frac{1}{\beta x^{\alpha-\beta}} = 0, \tag{8.4.16}$$

so

$$\lim_{x\to+\infty} \frac{\log(x)}{x^{\alpha}} = 0. \tag{8.4.17}$$

Q.E.D.

Exercise 8.4.2. Show that

$$\lim_{x\to 0^+} x^{\alpha}\log(x) = 0$$

for any rational number $\alpha > 0$.

8.5 The exponential function

Definition 8.5.1. We call the inverse of the logarithm function the *exponential* function. We denote the value of the exponential function at a real number x by $\exp(x)$.

Proposition 8.5.1. The exponential function has domain $\mathbb{R}$ and range $(0,+\infty)$. Moreover, the exponential function is increasing and differentiable on $\mathbb{R}$. If $f(x) = \exp(x)$, then $f'(x) = \exp(x)$.

Proof. Only the final statement of the proposition requires proof. If we let $g(x) = \log(x)$, then

$$f'(x) = \frac{1}{g'(\exp(x))} = \exp(x). \tag{8.5.1}$$

Q.E.D.

Proposition 8.5.2. For any real numbers x and y,

$$\exp(x+y) = \exp(x)\exp(y). \tag{8.5.2}$$

Proof. The result follows from

$$\log(\exp(x)\exp(y)) = \log(\exp(x)) + \log(\exp(y)) = x + y. \tag{8.5.3}$$

Q.E.D.

Proposition 8.5.3. For any real number x,

$$\exp(-x) = \frac{1}{\exp(x)}. \tag{8.5.4}$$

Proof. The result follows from

$$\log\left(\frac{1}{\exp(x)}\right) = -\log(\exp(x)) = -x. \tag{8.5.5}$$

Q.E.D.

Exercise 8.5.1. Use Taylor's theorem to show that

$$\exp(1) = e = \sum_{n=0}^{\infty} \frac{1}{n!}. \tag{8.5.6}$$

Proposition 8.5.4. For any rational number α,

$$\exp(\alpha) = e^{\alpha}. \tag{8.5.7}$$

Proof. Since $\log(e) = 1$, we have

$$\log(e^{\alpha}) = \alpha \log(e) = \alpha. \tag{8.5.8}$$

Q.E.D.

Definition 8.5.2. If α is an irrational number, we define

$$e^{\alpha} = \exp(\alpha). \tag{8.5.9}$$

Note that for any real numbers x and y,

$$e^{x+y} = e^x e^y \tag{8.5.10}$$

and

$$e^{-x} = \frac{1}{e^x}. \tag{8.5.11}$$

Moreover, $\log(e^x) = x$ and, if $x > 0$, $e^{\log(x)} = x$.

Definition 8.5.3. If x and a are real numbers with $a > 0$, we define

$$a^x = e^{x \log(a)}. \tag{8.5.12}$$

Exercise 8.5.2. Define $f : (0, +\infty) \to \mathbb{R}$ by $f(x) = x^a$, where $a \in \mathbb{R}$, $a \neq 0$. Show that $f'(x) = ax^{a-1}$.

Exercise 8.5.3. Suppose a is a positive real number and $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = a^x$. Show that $f'(x) = a^x \log(a)$.

Proposition 8.5.5. For any real number $\alpha > 0$,

$$\lim_{x \to +\infty} x^\alpha e^{-x} = 0. \tag{8.5.13}$$

Proof. We know that

$$\lim_{y \to +\infty} \frac{\log(y)}{y^{\frac{1}{\alpha}}} = 0. \tag{8.5.14}$$

Hence

$$\lim_{y \to +\infty} \frac{(\log(y))^\alpha}{y} = 0. \tag{8.5.15}$$

Letting $y = e^x$, we have

$$\lim_{x \to +\infty} \frac{x^\alpha}{e^x} = 0. \tag{8.5.16}$$

Q.E.D.

Proposition 8.5.6. For any real number α,

$$\lim_{x \to +\infty} \left(1 + \frac{\alpha}{x}\right)^x = e^\alpha. \tag{8.5.17}$$

Proof. First note that, letting $x = \frac{1}{h}$,

$$\lim_{x \to +\infty} \left(1 + \frac{\alpha}{x}\right)^x = \lim_{h \to 0^+} (1 + \alpha h)^{\frac{1}{h}} = \lim_{h \to 0^+} e^{\frac{1}{h} \log(1 + \alpha h)}. \tag{8.5.18}$$

Using l'Hôpital's rule, we have

$$\lim_{h \to 0^+} \frac{\log(1 + \alpha h)}{h} = \lim_{h \to 0^+} \frac{\alpha}{1 + \alpha h} = \alpha, \tag{8.5.19}$$

and the result follows from the continuity of the exponential function. **Q.E.D.**

Definition 8.5.4. We define the *hyperbolic sine* and *hyperbolic cosine* functions by

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \tag{8.5.20}$$

and

$$\cosh(x) = \frac{e^x + e^{-x}}{2}, \tag{8.5.21}$$

respectively.

Exercise 8.5.4. Show that for any real numbers x and y,

$$\sinh(x+y) = \sinh(x)\cosh(y) + \sinh(y)\cosh(x) \tag{8.5.22}$$

and

$$\cosh(x+y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y). \tag{8.5.23}$$

Exercise 8.5.5. Show that for any real number x,

$$\cosh^2(x) - \sinh^2(x) = 1. \tag{8.5.24}$$

Exercise 8.5.6. If $f(x) = \sinh(x)$ and $g(x) = \cosh(x)$, show that

$$f'(x) = \cosh(x) \tag{8.5.25}$$

and

$$g'(x) = \sinh(x). \tag{8.5.26}$$

Index